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Separability generalizes Dirac's theorem

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Abstract

In our study of the extremities of a graph, we define a moplex as a maximal clique module the neighborhood of which is a minimal separator of the graph. This notion enables us to strengthen Dirac's theorem (Dirac, 1961): "*Every non-clique triangulated graph has at least two non-adjacent simplicial vertices*", restricting the definition of a simplicial vertex; this also enables us to strengthen Fulkerson and Gross' simplicial elimination scheme; thus provides a new characterization for triangulated graphs.

Our version of Dirac's theorem generalizes from the class of triangulated graphs to any undirected graph: "*Every non-clique graph has at least two non-adjacent moplexes*".

To insure a linear-time access to a moplex in any graph, we use an algorithm due to Rose Tarjan and Lueker (1976) for the recognition of triangulated graphs, known as LexBFS: we prove a new invariant for this, true even on non-triangulated graphs. © 1998 Elsevier Science B.V. All rights reserved.

Keywords: Simplicial vertex; Perfect elimination ordering; Minimal triangulation; LexBFS; Minimal separator; Moplex

1. Introduction

Our work is based on the notion that the *maximal clique modules* of a graph in many respects behave as a single vertex, and must be taken into account when one wants to extract elimination or composition schemes, or neighborhood properties. '*Belonging to the same maximal clique module*' defines an equivalence relation, and a maximal clique module is just a vertex of the corresponding quotient graph.

The notion of maximal clique module was first implicitly used by Roberts [14], as necessary to obtain a unique representation of proper interval graphs, but is not taken up again on more general graphs. Although he did not express it in this fashion, Roberts defined the equivalence relation on vertices $x \sim y$ iff x belongs to the same

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maximal clique module as y, so as to work on the quotient graph, which has the desired unique intervallary representation.

Our research was motivated by understanding the structure of the minimal separators of a graph, so we use a special kind of maximal clique module: we define a '*maximal clique module whose neighborhood is a minimal separator of the graph*' (we call this object a '*moplex*' for short).

This leads us to a generalization of Dirac's theorem for triangulated graphs from "*Every non-clique triangulated graph has at least two non-adjacent simplicial vertices.*" to: "*Every non-clique graph has at least two non-adjacent moplexes*".

The main contribution of this paper is a general invariant for Lex BFS: at any step, the algorithm terminates on a vertex belonging to a moplex. Many have tried before to draw a parallel between LexBFS and minimal separation, but failed because when considering a vertex which does not belong to a trivial maximal clique module, its neighborhood is a separator but not a minimal one, as the neighborhood contains noise (i.e. the other members the maximal clique module).

2. Notations and previous results

We will denote set inclusion by $\subseteq$, and strict inclusion by $\subset$.

2.1. Graphs

$G=(V,E)$ is a finite undirected graph with vertex set V and edge set E, $|V|=n$, $|E|=m$. A 4-*cycle* is a chordless cycle on four vertices. A *clique* is a set of pairwise adjacent vertices. $N(x)$ denotes the neighborhood of vertex x (it does not contain x), $N(X)$ is the neighborhood of $X\subseteq V$: $N(X)=(\bigcup_{x\in X}N(x))\backslash X$. $N_H(X)$ is the neighborhood of X in graph H. We say that a vertex x *sees* another vertex y if x and y are adjacent, and that x sees $A\subseteq V$ iff there is some vertex of A that x sees. A vertex x is *simplicial* iff $N(x)$ is a clique. The *deficiency* of a vertex set X is $\mathrm{Def}(X)=\{\{a,b\}\in N(X)|\, ab\notin E\}$. We will write $G'\leftarrow G+\mathrm{Def}(X)$ to describe the addition to G of the edges necessary to make $N(X)$ into a clique.

A *module* is a subset A of V such that: $(\forall\ a_i,a_j\in A,\ N(a_i)\cap N(A)=N(a_j)\cap N(A))$. We will call a single vertex a *trivial module*. We will call $A\subset V$ a maximal-clique-module iff A is a module and a clique, and is inclusion-maximal for both properties.

2.2. Separation

A subset $S\subseteq V$ of a connected graph G is called a *separator* iff $G(V\backslash S)$ is disconnected. This defines a set (of size $\geqslant 2$) of connected components, denoted $\mathrm{CC}(S)$ (components are vertex sets). A component F in $\mathrm{CC}(S)$ is called a *full component* iff $N(F)=S$. S is called an *ab-separator* iff a and b lie in two different components of

CC(S). S is an ab-separator iff every path from a to b intersects S. S is called a *minimal ab-separator* iff S is an ab-separator and no proper subset of S is also an ab-separator. S is a minimal ab-separator iff a and b lie in two different full components of CC(S). S is called a *minimal separator* iff $\exists\ a, b \in V$ such that S is a minimal ab-separator.

2.3. Triangulated graphs

A graph G is *triangulated* iff it contains no chordless cycle of length greater than 3. Dirac showed (see [6]) that a graph is triangulated iff every minimal separator is a clique (Dirac's theorem), and that every non-clique triangulated graph has at least two non-adjacent simplicial vertices (Dirac's characterization).

Confluence Lemma. *Let H be a triangulated graph, S a minimal separator of H; in any full component F of* CC(S), *there is some vertex $a \in F$, called a confluence vertex, which is adjacent to every vertex of S.*

Fulkerson and Gross [7] suggested to use Dirac's work for the following characterization of triangulated graphs: *G is triangulated iff one can repeatedly find a simplicial vertex and delete it from the graph, until no vertex remains.* This is called a *simplicial elimination scheme*, and defines a linear ordering α on the vertices of the graph, called a *perfect elimination ordering* (*peo*).

2.4. Triangulation (see [13, 15])

A triangulated graph $H = (V, E + F)$, is called a *triangulation* of $G = (V, E)$.

The triangulation is said to be *minimal* iff for any edge e of F, $H' = (V, (E \cup F) \backslash e)$ is not triangulated. F is then called a *minimal fill-in.*

Unique chord property [15]: *A triangulation H of G is minimal iff $\forall e \in F$, e is the unique chord of some* 4-*cycle of H.*

Crossing edge lemma [17]: *No edge of a minimal fill-in of G can join two connected components of a clique separator of G.*

Property 2.1. *Let H be a minimal triangulation of G. Any minimal ab-separator of H is also a minimal ab-separator of G.*

2.5. Interval graphs

A graph $G = (V, E)$ is an *interval graph* iff there is an assignment to each vertex $x \in V$ of an interval $J(x)$ on the real line such that $x, y \in E \Leftrightarrow J(x) \cap J(y) \neq \emptyset$. A *proper interval graph* (also called indifference graph, see [14]) is an interval graph which allows a representation by a family of intervals such that no interval properly contains another.

3. Moplexes in triangulated graphs

The notion of extreme point was introduced by Roberts (see [14]) to characterize the two vertices which are extremal in the representation of a proper interval graph. He defined the following equivalence relation: $x \sim y$ iff $((\forall z \in V)(xz \in E \text{ iff } yz \in E))$: the quotient graph $G^* = G \backslash \sim$ had the desired property. Translated into graph notions, on a reflexive graph vertices x and y are equivalent iff they belong to the same maximal clique module.

Dirac, in his study of triangulated graphs (see [6]), showed that "*Every non-clique triangulated graph has at least two non-adjacent simplicial vertices*". We use Robert's equivalence relation, together with separability considerations, to strengthen Dirac's theorem.

Definition 3.1. We will call $A \subset V$ a maximal clique module *iff* A is both a module and a clique, and A is inclusion-maximal for both properties.

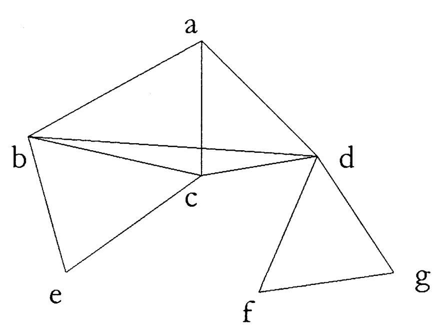
Definition 3.2
We call *moplex* a maximal clique module whose neighborhood is a minimal separator.
We will say that a moplex is *simplicial* iff its neighborhood is a clique.
We will say that a moplex is *trivial* iff it has only one vertex.

Property 3.3. *Let H be a triangulated graph. Every moplex M is simplicial, and every vertex of M is a simplicial vertex.*

Proof. Let H be a triangulated graph, let M be a moplex of H. By definition, $N(M)$ is a minimal separator. By Dirac's characterization, $N(M)$ is a clique; $M \cup N(M)$ must be a clique. Let $x \in M$; $N(x) \subseteq M \cup N(M)$ must be a clique, and x is simplicial. □

Remark 3.4. The converse is not true: in a triangulated graph, a vertex may well be simplicial, without belonging to any moplex (see Example 3.5).

Example 3.5



This triangulated graph has two moplexes: $\{e\}$ and $\{f, g\}$; a, e, f and g are all simplicial, but a does not belong to a moplex

Theorem 3.6. *Any non-clique triangulated graph has at least two non-adjacent simplicial moplexes.*

Proof. We use Dirac's original proof scheme, replacing "vertex" with "moplex".

By induction: For $n = 3$, the only non-clique graph is a P_3 abc. The only minimal separator is $\{b\}$, and there are 2 trivial moplexes, a and c.

Since G is not a clique, it has at least one minimal separator. Since G is triangulated, S is a clique. Let A and B be two full components of CC(S).

1. If $A \cup S$ is a clique, then $N(A) = S$. A is both a module and a clique (else $A \cup S$ cannot be a clique). A is maximal as a clique-module, because we cannot enlarge it: if we forgot a vertex s to make A into a maximal clique module, we must take s in $N(A)$; but since $N(A)$ is a minimal separator, s must see some $b \in B$, but b cannot see A, as b and each vertex of A belong to different connected components. S is a minimal separator, so A is a moplex.

2. $A \cup S$ is not a clique: by induction hypothesis $A \cup S$ has two non-adjacent moplexes. At least one of them (call it M) is not in S (as they are non-adjacent). $N(M)$ is the same in $A \cup S$ as in G. Thus $N(M)$ is a minimal separator for G as well as for $A \cup S$. Similarly, $B \cup S$ also yields a moplex. □

Note that actually Dirac shows that all the connected components (not just the full ones) yield a simplicial vertex; we could likewise extend our proof.

In Example 3.5, the set of minimal separators is: $\{\{b, c\}, \{d\}\}$. There are two moplexes: $\{e\}$ and $\{f, g\}$.

Theorem 3.6 is a strengthening of Dirac's theorem because of Remark 3.4. In the same fashion, we strengthen Fulkerson and Gross' characterizing elimination scheme for triangulated graphs ([7]): "*A graph is triangulated iff one can repeatedly delete a simplicial vertex until nothing remains*".

Characterization 3.7. *A graph is triangulated iff one can repeatedly delete a simplicial moplex until the graph is a clique.*

Proof. ($\Rightarrow$) Let G be a triangulated graph. By Theorem 3.6, it has a moplex X, which is simplicial by property 3.3. X can be eliminated; $G \backslash X$ is still a triangulated graph.

($\Leftarrow$) Any simplicial elimination scheme on moplexes is a simplicial elimination scheme, because by Property 3.3 a moplex contains only simplicial vertices. Thus any graph with such a scheme must be triangulated. □

Remark. Applied to Robert's work, Theorem 3.6 yields a new characterization of the extreme points of a proper interval graph: a vertex of G^* is extremal iff it corresponds to a moplex of G.

4. Generalization of Dirac's theorem to any graph

The strengthened version of Dirac's theorem for triangulated graphs (Theorem 3.6) generalizes easily to a non-triangulated graph.

Main Theorem 4.1. *Any non-clique graph has at least two non-adjacent moplexes.*

To prove this, we will show that if H is a minimal triangulation of G, and A a moplex of H, then A is a moplex of G.

Lemma 4.2. *Let H be a minimal triangulation of G, let A be a moplex of H. Then $N_H(A) = N_G(A)$.*

Proof. Let A be a moplex of H. Since a minimal triangulation is obtained by adding edges without deleting any, $N_G(A) \subseteq N_H(A)$. Suppose $N_G(A) \neq N_H(A)$. Take z in $N_H(A) \backslash N_G(A)$, a in A. Since H is a minimal triangulation of G, by the unique chord property, az must be the unique chord of some 4-cycle in H: $axzya$. But then $x, y \in N_H(A)$, which is a clique, being a minimal separator of a triangulated graph: x must be adjacent to y and az cannot be the unique chord. □

Lemma 4.3. *Let $H = (V, E + F)$ be a minimal triangulation of $G = (V, E)$, let A be a moplex of H. A is a moplex of G.*

Proof. Let A be a moplex of H. Let $N(A) = N_H(A) = N_G(A)$. $A \cup N(A)$ is a clique of H. Let us show, that A is a moplex of G.

We will show that A is a clique of G: suppose $\exists\, a, b \in A$ such that $a \notin N_G(b)$: ab must belong to F, so ab must be the unique chord of some 4-cycle $axbya$ of H. But in H, x must see y, as they are both neighbors of a and $A \cup N(A)$ is a clique: ab cannot be the unique chord of $axbya$.

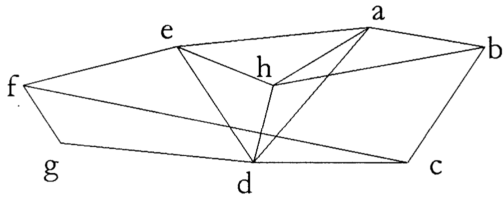
A is a module of G: if not, there is some vertex z in $N(A)$ which fails to see some vertex a of A in G. Edge az must be in F, but this is again impossible because of the unique chord property.

A is maximal in G: suppose we forgot some vertex s: we must take s in $N(A)$. $N(A)$ is a minimal separator of G by property 2.1. s must see B, the second full component of $N(A)$, and B cannot see A. □

Proof of Main Theorem 4.1. Let G be a graph that is not a clique. Let H be a minimal triangulation of G. By Theorem 3.6, H has at least two non-adjacent moplexes, A and B. By Lemma 4.3, A and B are also moplexes of G. □

Note that theorem 4.1 generalizes Berge's theorem [1]: "Every tree has at least 2 leaves". In a tree, being a moplex is equivalent to being a leaf.

Example. Graph G with set of minimal separators $\{\{a, c, h\}, \{d, f\}, \{a, d, f, h\}, \{b, d, e\}, \{b, d, f\}, \{c, d, e\}, \{c, e, g\}\}$. Set of moplexes: $\{\{a, h\}, b, c, e, f, g\}$.



5. Algorithm LexBFS

We now prove that a moplex can be found in linear time. We use Rose, Tarjan and Lueker's [15] beautiful linear-time algorithm for the recognition of triangulated graphs, known as Lexico Breadth-First Search (LexBFS). We show that LexBFS ends on a moplex in any graph.

Algorithm LexBFS
Input: a graph $G = (V, E)$
Output: an ordering α of the vertices

Initialize all labels as empty on all vertices.

For $i = n$ downto 1 step -1 do
1. pick an unnumbered vertex x with largest label and assign x the number i: $\alpha(i) \leftarrow x$.
2. for each unnumbered vertex $a \in N(x)$ do
   add i to label(a)

Remark. We will denote by α^{-1} the inverse of α: $\alpha(i) = x$ iff $\alpha^{-1}(x) = i$. Order on labels is dictionary order. (see [8].)

Step 1 insures that every neighbor of a numbered vertex will be chosen before any as yet unnumbered vertex.

Step 2 insures that an unnumbered vertex inherits a larger score if it has many high-numbered neighbors.

If label(a) < label(b) at any time during the execution of LexBFS, then this stays true until a is numbered (and a has to be numbered before b).

Theorem 5.1. *LexBFS ends on a vertex which belongs to a moplex.*

To prove this theorem, we will need several lemmas. We will also need some notations to describe what happens in the vicinity of the minimal separator defined by the moplex LexBFS ends on.

We will say that LexBFS "starts" on vertex $\alpha(n)$ and "ends" on vertex $\alpha(1)$: because the numbering takes place in decreasing order, a is numbered before b iff $\alpha^{-1}(a) > \alpha^{-1}(b)$. This is a little confusing, but essential, because in a triangulated graph LexBFS yields a perfect elimination ordering: vertex number 1 is simplicial, and if it is eliminated, vertex number 2 will be simplicial in the resulting graph, etc.

Lemma 5.2. *Let α be the ordering produced by an execution of LexBFS on a non-clique graph; unless $\alpha(1)$ is universal, $\alpha(1)$ cannot be adjacent to $\alpha(n)$.*

Proof. If $\alpha(n)$ is universal, all vertices of $G\backslash\{\alpha(n)\}$ have the same label just after a is numbered, and LexBFS would run the same way in $G\backslash\{\alpha(n)\}$ as in G. Since G is not a clique, there is some non-empty subgraph with no universal vertex to start with.

If $\alpha(n)$ is not universal, there is some vertex y that is unlabeled at the time $\alpha(1)$ is labeled by $\alpha(n)$: $\alpha(1)$ will be numbered before y, and $\alpha(1)$ cannot be numbered last. □

Let X be the maximal clique module $\alpha(1)$ belongs to. We want to prove that $N(X)$ is a minimal separator of the graph, but trivially it is a separator (it separates vertex $\alpha(1)$ from $\alpha(n)$). We will denote the set of connected components it defines $\mathrm{CC}(N(X)) = \{C_1, C_2, \dots, C_k, X\}$ in the order they are found and numbered by LexBFS: thus C_1 is the component which contains $\alpha(n)$. Note that $N(C_i) \subseteq N(\alpha(1))$: $N(C_i)$ is just the set of vertices of $N(X)$ which see component C_i.

Lemma 5.3. *Let C_1 be the component of $\mathrm{CC}(N(X))$ which contains $\alpha(n)$. No vertex of $G\backslash(C_1 \cup N(C_1))$ can be numbered before all vertices of $(C_1 \cup N(C_1))$ are numbered.*

Proof. Let b be the first vertex of $G\backslash(C_1 \cup N(C_1))$ to be numbered.
- Suppose there is some vertex v in $G\backslash(C_1 \cup N(C_1))$ such that $\alpha(b) > \alpha(v)$.
- At the time b is about to be numbered, label $(b) \geqslant \mathrm{label}(\alpha(1))$. But since b does not see C_1 and $\alpha(1)$ sees all of $N(C_1)$, the only numbered neighbors of b are neighbors of $\alpha(1)$: $\mathrm{label}(\alpha(1)) \geqslant \mathrm{label}(b)$. Thus $\mathrm{label}(b) = \mathrm{label}(\alpha(1))$.

1. Suppose $v \in C_1$.

Since C_1 is connected, there must be a path μ from a to v in C_1. μ contains both numbered vertices and unnumbered vertices. Let z be the first unnumbered vertex of μ.

Suppose $\mathrm{label}(b) \geqslant \mathrm{label}(z)$. Because z inherits from its predecessor in μ, $\mathrm{label}(b) \neq \mathrm{label}(z)$. Thus we must have $\mathrm{label}(b) > \mathrm{label}(z)$. But $\mathrm{label}(\alpha(1)) = \mathrm{label}(b)$, thus $\mathrm{label}(x) > \mathrm{label}(z)$, and $\alpha(1)$ could not be numbered last.

Therefore there can be no unnumbered vertex in C_1 at the time b is chosen.

2. Suppose $v \in N(C_1)$ (v is thus in $N(\alpha(1))$); $\mathrm{label}(b) \geqslant \mathrm{label}(v)$. v sees C_1 and b does not, and as we have just said, C_1 must be completely numbered: $\mathrm{label}(b) \neq \mathrm{label}(v)$. But again $\mathrm{label}(b) = \mathrm{label}(\alpha(1)) > \mathrm{label}(v)$, and $\alpha(1)$ could not be numbered last. □

Lemma 5.4. *When LexBFS starts numbering $G\backslash(C_1 \cup N(C_1))$, every unnumbered vertex has exactly the same label.*

Proof. We will denote by λ the label consisting of all vertices of $N(C_1)$ taken in decreasing order.

When LexBFS starts numbering $G\backslash(C_1 \cup N(C_1))$, label$(\alpha(1)) = \lambda$. Suppose there is some vertex b in $G\backslash(C_1 \cup N(C_1))$ such that label$(b) \neq$ label$(\alpha(1))$. label(b) > label$(\alpha(1))$. There must be some numbered vertex which b inherited from but not $\alpha(1)$; b cannot see C_1, so the only numbered vertices b can see must be in $N(C_1) \subset$ N$(\alpha(1))$.

Thus label$(b) =$ label$(\alpha(1))$ $\forall b \in G\backslash(C_1 \cup N(C_1))$. □

Property 5.5. *Let X be the maximal clique module $\alpha(1)$ belongs to, let $k = |X|$. The vertices of X receive numbers* 1 *through k.*

Proof of Theorem 5.1 and of Property 5.5. Let X be the maximal clique module $\alpha(1)$ belongs to. We claim that X is a moplex.

By Lemma 5.3, all vertices of $C_1 \cup N(C_1)$ are numbered before any other. By Lemma 5.4, when LexBFS starts numbering $G\backslash(C_1 \cup N(C_1))$, all vertices have the same label, thus the restriction of LexBFS to $G\backslash(C_1 \cup N(C_1))$ will also be a LexBFS execution. Because of Lemma 5.3, we must proceed with a vertex of $C_2 \cup N(C_2)$, and by Lemma 5.2 we must begin in C_2. Recursively, LexBFS numbers $C_1 \cup N(C_1)$, then $C_2 \cup N(C_2)$, ... then $C_k \cup N(C_k)$. At the time all vertices of $C_k \cup N(C_k)$ have just been numbered, all unnumbered vertices have the same label.

Let $A = G\backslash(C_1 \cup N(C_1) \cup C_2 \cup N(C_2) \cup \cdots \cup C_k \cup N(C_k))$. By what we describe above, A is what is left unnumbered when LexBFS has finished numbering $C_k \cup N(C_k)$, and all vertices of A have the same label. We claim that $A = X$.

X is a component of $N(X)$. By Lemma 5.3, if one vertex of X is labeled before we start numbering A, then all the vertices of X must likewise be labeled: $\alpha(1)$, which is in X, could not be numbered last; this ensures that $X \subseteq$ A.

Clearly, $X \subseteq A \subseteq X \cup N$(X).

Considering the restriction of LexBFS to A, by Lemma 5.2, A is a clique. Since all vertices of A have the same label, A is a module. A is a module and a clique and contains X, a maximal clique module: $A = X$.

Since $A = X$ and the vertices of A are all that are left unnumbered, all the vertices of X are numbered consecutively.

We claim that the last component C_k is a full component of $N(X)$: $\forall i < k$, every vertex of $N(C_i)$ must see every vertex c of C_k (else label$(\alpha(1)) >$ label(c), and $\alpha(1)$ could not be numbered last). Thus every vertex of $N(X)$ sees C_k, so $N(C_k) = N(X)$: by definition, C_k is a full component of CC$(N(X))$. Since X is also a full component, $N(X)$ is a minimal separator and X is a moplex. □

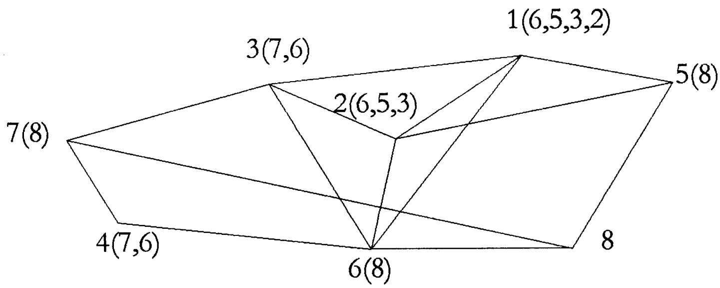
Note that LexBFS labels the vertices of X in such a fashion that we can delimit it with no extra cost: for linear-time implementation, LexBFS is built upon a partitioning process of V (see [8]). At each step, the vertices bearing the same label form a class of this partition. The last vertex v of the last class is chosen and eliminated; if at each step the current vertex is preserved in the class of its neighbors, the moplex X is exactly the first class of the resulting partition.

Algorithmic invariant of LexBFS 5.6. *At each step, LexBFS chooses a vertex which belongs to a moplex of the subgraph induced by the numbered vertices.*

Proof. Immediate by Theorem 5.1. □

Remark 5.7. By Property 5.3, in a triangulated graph, every vertex belonging to a moplex is simplicial: our invariant yields a direct proof for LexBFS on a triangulated graph.

Example 5.8. A LexBFS execution on a non-triangulated graph; set of minimal separators: $S = \{\{1,2,8\}, \{6,7\}, \{1,2,6,7\}, \{3,5,6\}, \{5,6,7\}, \{3,6,8\}, \{3,4,8\}\}$; set of moplexes: $\{\{1,2\}, 5,8,3,7,4\}$. This execution ends on moplex $\{1,2\}$.



Remark. We conjecture that LexBFS's cousin MCS (see[18]) ends on a moplex in any triangulated graph, although it is not the case for a non-triangulated graph.

Recently, Corneil, et al. [4] showed that a double execution of LexBFS yields a pokable dominating pair in an AT-Free graph (this applies to interval graphs and to proper interval graphs, which are subclasses of AT-Free graphs). This further illustrates how moplexes can be considered as extremities of a graph.

LexBFS yields a special kind of moplex, with strong properties: Dirac's theorem could be further strengthened accordingly, as a double execution of LexBFS always yields a pair of such moplexes. A characterization of these would be very interesting.

6. Conclusion

It is probable that many generalizations that have been made of Dirac's theorem to classes of perfect graphs (see for instance, [5, 9–11]), are related to our generalization.

We also feel that our contribution helps to explain why LexBFS is so powerful, even on non-triangulated graphs.

Moplexes seem to be a new and important concept in graph theory. In particular, we apply them to a question left open by Tarjan [17] as to the existence of a unique graph decomposition by clique separators [3]: using moplexes instead of vertices enables us to ensure a decomposition by clique separators that are also minimal separators of the graph; this decomposition is unique and decomposes the graph into a minimum number of atoms. Moreover, we can use our LexBFS algorithmic invariant to do so with the same worst-time complexity as Tarjan's.

Aknowledgements

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