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Screw theory (torsor theory)
Vector and pseudo-vector representations, twist, wrench

Gilles Leborgne

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A screw (also called a torsor) is an antisymmetric vector field in a Euclidean setting. It is called a twist (or a kinematic screw, or a distributor) when it is the velocity field of a rigid body motion, and called a wrench when it is the moment of a force field.

To avoid confusions and misunderstandings, the first paragraphs are devoted to the definitions of vectors, pseudo-vectors, vector products, pseudo-vector products, antisymmetric endomorphisms and their representations.

Contents

The notation $g := f$ means: f being given, g is defined by $g = f$.

1 Dimension 3 vector spaces

1.1 The theoretical vector space $\overrightarrow{\mathbb{R}^3}$ and the different $\overrightarrow{\mathbb{R}^3}$ in mechanics

1.1.1 Dimension 3 vector space

A dimension 3 real vector space $V = (V, +, .)$ is a theoretical vector space (mathematical model) where "+" is an internal operation and "." an external operation s.t. $(V, +)$ is a commutative group, and $\lambda.\vec{v} \in V$ for all $\lambda \in \mathbb{R}$ and $\vec{v} \in V$, with the usual distributivity rules (with 1 the unitary element in $\mathbb{R}$): $1.\vec{v} = \vec{v}$, $\lambda.(\vec{v} + \vec{w}) = \lambda.\vec{v} + \lambda.\vec{w}$, $(\lambda + \mu).\vec{v} = \lambda.\vec{v} + \mu.\vec{v}$, and $(\lambda\mu).\vec{v} = \lambda.(\mu.\vec{v})$, for all $\lambda, \mu \in \mathbb{R}$ and $\vec{v}, \vec{w} \in V$.

If a basis $(\vec{a}_i)_{i=1,2,3} \overset{\text{noted}}{=} (\vec{a}_i)$ in V is chosen, then a vector $\vec{v} = \sum_{i=1}^3 v_i\vec{a}_i \in V$ is represented by the column matrix $[\vec{v}]_{\vec{a}} := \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$. And writing $[\vec{v}]_{\vec{a}} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ means $\vec{v} \in V$ and $\vec{v} = \sum_{i=1}^3 v_i\vec{a}_i$. A bilinear form $z(\cdot,\cdot) : V \times V \to \mathbb{R}$ (e.g. a scalar dot product) is represented by the $3 * 3$ matrix $[z]_{\vec{a}} := [z(\vec{a}_i, \vec{a}_j)]_{\substack{i=1,2,3 \\ j=1,2,3}} \overset{\text{noted}}{=} [z(\vec{a}_i, \vec{a}_j)]$. Thus, with $\vec{v} = \sum_{i=1}^3 v_i\vec{a}_i$ and $\vec{w} = \sum_{i=1}^3 w_i\vec{a}_i$ in V, the bilinearity of $z(\cdot,\cdot)$ gives $z(\vec{v}, \vec{w}) = \sum_{i,j=1}^3 v_i w_j z(\vec{a}_i, \vec{a}_j) = [\vec{v}]_{\vec{a}}^T.[z]_{\vec{a}}.[\vec{w}]_{\vec{a}}$.

1.1.2 Our usual affine space $\mathbb{R}^3$, and $\overrightarrow{\mathbb{R}^3}$ and Euclidean setting

Affine setting: $\mathbb{R}^3$ is our usual affine space of points, and $\overrightarrow{\mathbb{R}^3}$ is its associated vector space made of the "bi-point" vectors $\overrightarrow{AB} \overset{\text{noted}}{=} B - A$ for all $A, B \in \mathbb{R}^3$, in which case we write $B = A + \overrightarrow{AB}$.

Euclidean setting: Choose a unit of measure of length u, e.g. the metre or the foot, to be able to build a Euclidean basis $(\vec{e}_i)_{i=1,2,3} \overset{\text{noted}}{=} (\vec{e}_i)$ in $\overrightarrow{\mathbb{R}^3}$: The length of each $\vec{e}_i$ is 1 in the unit u, and the length of $3\vec{e}_i + 4\vec{e}_{i+1}$ is 5 (Pythagoras orthogonality) in the unit u, for all $i = 1, 2, 3$, where $\vec{e}_4 := \vec{e}_1$.

The Euclidean dot product $e(\cdot,\cdot) : \overrightarrow{\mathbb{R}^3} \times \overrightarrow{\mathbb{R}^3} \to \mathbb{R}$ associated to $(\vec{e}_i)$ is the symmetric definite positive bilinear form defined by, for all $\vec{v} = \sum_{i=1}^3 v_i\vec{e}_i$ and $\vec{w} = \sum_{i=1}^3 w_i\vec{e}_i$ in $\overrightarrow{\mathbb{R}^3}$,

$$e(\vec{v}, \vec{w}) = \sum_{i,j=1}^3 v_i w_i = [\vec{v}]_{\vec{e}}^T.[\vec{w}]_{\vec{e}} \overset{\text{noted}}{=} (\vec{v}, \vec{w})_e \overset{\text{noted}}{=} \vec{v} \bullet_e \vec{w}, \tag{1.1}$$

i.e. defined by $[e]_{\vec{e}} = I$ (identity matrix) (i.e. $e(\vec{e}_i, \vec{e}_j) = \delta_{ij} = (\vec{e}_i, \vec{e}_j)_e = \vec{e}_i \bullet_e \vec{e}_j$, for all $i, j = 1, 2, 3$). The associated Euclidean norm $||.||_e$ is given by $||\vec{v}||_e := \sqrt{\vec{v} \bullet_e \vec{v}} = \sqrt{(\vec{v}, \vec{v})_e}(= \sum_{i,j=1}^3 v_i^2$ when $\vec{v} = \sum_{i=1}^3 v_i\vec{e}_i)$.

And two vectors $\vec{v}, \vec{w} \in \overrightarrow{\mathbb{R}^3}$ are Euclidean orthogonal iff $\vec{v} \bullet_e \vec{w} = 0$.

1.1.3 The different $\overrightarrow{\mathbb{R}^3}$ in mechanics

In mechanics we need "a compatible dimension" for a sum to be defined: You don't add velocities with accelerations or with forces or with moments... Thus we define several dimension 3 real vector spaces V corresponding to different dimensions: V_{vel} for the velocities, V_{acc} for accelerations, V_{for} for the forces, V_{mom} for moments...

However, when a unit of measure of length u is chosen, and systematically used by all observers, all the dimension 3 spaces using a length are abusively called $\overrightarrow{\mathbb{R}^3}$: $V_{vel} \overset{\text{noted}}{=} \overrightarrow{\mathbb{R}^3}$, $V_{acc} \overset{\text{noted}}{=} \overrightarrow{\mathbb{R}^3}$, $V_{for} \overset{\text{noted}}{=} \overrightarrow{\mathbb{R}^3}$, $V_{mom} \overset{\text{noted}}{=} \overrightarrow{\mathbb{R}^3}$... (for operations relative to measures of length). This is the case for calculations relative to screws (torsors).

1.2 The vector product associated with a basis

Definition 1.1 The vector product associated with a basis $(\vec{a}_i)$ in a dimension 3 real vector space V is the bilinear antisymmetric map $\times_a(\cdot,\cdot) : V \times V \to V$ defined by, for all $\vec{v} = \sum_{i=1}^3 v_i\vec{a}_i$ and $\vec{w} = \sum_{i=1}^3 w_i\vec{a}_i$ in V,

$$\times_a(\vec{v}, \vec{w}) := (v_2w_3 - v_3w_2)\vec{a}_1 + (v_3w_1 - v_1w_3)\vec{a}_2 + (v_1w_2 - v_2w_1)\vec{a}_3 \overset{\text{noted}}{=} \vec{v} \times_a \vec{w}, \tag{1.2}$$

i.e.

$$[\vec{v} \times_a \vec{w}]_{\vec{a}} := \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix} \overset{\text{noted}}{=} \det(\begin{pmatrix} \vec{a}_1 & v_1 & w_1 \\ \vec{a}_2 & v_2 & w_2 \\ \vec{a}_3 & v_3 & w_3 \end{pmatrix}), \tag{1.3}$$

the formal determinant being expanded along the first column. ($\vec{v} \times_a \vec{w}$ is written $\vec{v} \wedge_a \vec{w}$ in French.) In other words, $\times_a$ is defined by $\vec{a}_i \times_a \vec{a}_{i+1} = \vec{a}_{i+2}$ for $i = 1, 2, 3$, where $\vec{a}_4 := \vec{a}_1$ and $\vec{a}_5 := \vec{a}_2$.

We immediately check: $\times_a$ is indeed bilinear and antisymmetric ($\vec{w} \times_a \vec{v} = -\vec{v} \times_a \vec{w}$).

Exercise 1.2 Define the basis $(\vec{b}_i)$ by $\vec{b}_1 = -\vec{a}_1$, $\vec{b}_2 = \vec{a}_2$, $\vec{b}_3 = \vec{a}_3$ (change of orientation, drawing). Prove:

$$\times_b = -\times_a, \quad \text{i.e.} \quad \vec{v} \times_b \vec{w} = -\vec{v} \times_a \vec{w}, \ \ \forall \vec{v}, \vec{w} \in V. \tag{1.4}$$

(The definition of a vector product is basis dependent.)

Answer. $\vec{b}_2 \times_b \vec{b}_3 = \vec{b}_1 = -\vec{a}_1 = -\vec{a}_2 \times_a \vec{a}_3 = -\vec{b}_2 \times_a \vec{b}_3$, and $\vec{b}_3 \times_b \vec{b}_1 = \vec{b}_2 = \vec{a}_2 = \vec{a}_3 \times_a \vec{a}_1 = -\vec{b}_3 \times_a \vec{b}_1$, and $\vec{b}_1 \times_b \vec{b}_2 = \vec{b}_3 = \vec{a}_3 = \vec{a}_1 \times_a \vec{a}_2 = -\vec{b}_1 \times_a \vec{b}_2$; And $\times_a$ and $\times_b$ are bilinear antisymmetric, hence (1.4). ▪

Exercise 1.3 Let $(\cdot,\cdot)_a \overset{\text{noted}}{=} . \bullet_a .$ be the dot product associated to $(\vec{a}_i)$, i.e. defined by $(\vec{a}_i, \vec{a}_j)_a = \delta_{ij} = \vec{a}_i \bullet_a \vec{a}_j$ for all i, j. Check:

$$\vec{u} \times_a (\vec{v} \times_a \vec{w}) = (\vec{u} \bullet_a \vec{w})\vec{v} - (\vec{u} \bullet_a \vec{v})\vec{w}. \tag{1.5}$$

Answer. $[\vec{u} \times_a (\vec{v} \times_a \vec{w})]_{\vec{a}} = \begin{pmatrix} u_2(v_1w_2 - v_2w_1) - u_3(v_3w_1 - v_1w_3) \\ u_3(v_2w_3 - v_3w_2) - u_1(v_1w_2 - v_2w_1) \\ u_1(v_3w_1 - v_1w_3) - u_2(v_2w_3 - v_3w_2) \end{pmatrix} = \begin{pmatrix} (\sum_{i=1}^3 u_iw_i)v_1 - (\sum_{i=1}^3 u_iv_i)w_1 \\ (\sum_{i=1}^3 u_iw_i)v_2 - (\sum_{i=1}^3 u_iv_i)w_2 \\ (\sum_{i=1}^3 u_iw_i)v_3 - (\sum_{i=1}^3 u_iv_i)w_3 \end{pmatrix}$. ▪

1.3 Determinant associated with a basis

Definition 1.4 The determinant associated with a basis $(\vec{a}_i)$ is the tri-linear alternated form $\det_{\vec{a}} : V \times V \times V \to \mathbb{R}$ defined by

$$\det_{\vec{a}}(\vec{a}_1, \vec{a}_2, \vec{a}_3) = +1, \tag{1.6}$$

i.e. defined by, for all $\vec{u} = \sum_{i=1}^3 u_i\vec{a}_i$, $\vec{v} = \sum_{i=1}^3 v_i\vec{a}_i$ and $\vec{w} = \sum_{i=1}^3 w_i\vec{a}_i$ in V,

$$\det_{\vec{a}}(\vec{u}, \vec{v}, \vec{w}) = u_1(v_2w_3 - v_3w_2) + u_2(v_3w_1 - v_1w_3) + u_3(v_1w_2 - v_2w_1). \tag{1.7}$$

In other words, with $\det_M(\mathcal{A})$ the determinant of a $3*3$ matrix $\mathcal{A}$,

$$\det_{\vec{a}}(\vec{u}, \vec{v}, \vec{w}) := \det_M(\, [\vec{u}]_{\vec{a}} \quad [\vec{v}]_{\vec{a}} \quad [\vec{w}]_{\vec{a}} \,), \tag{1.8}$$

where $(\, [\vec{u}]_{\vec{a}} \quad [\vec{v}]_{\vec{a}} \quad [\vec{w}]_{\vec{a}} \,)$ is the $3*3$ matrix made of the columns $[\vec{u}]_{\vec{a}}$, $[\vec{v}]_{\vec{a}}$, $[\vec{w}]_{\vec{a}}$.

Remark: When $(\vec{a}_i)$ is a Euclidean basis, $\det_{\vec{a}}(\vec{u}, \vec{v}, \vec{w})$ is the algebraic volume (or signed volume) of the parallelepiped limited by the three vectors $\vec{u}, \vec{v}, \vec{w}$ (in the chosen unit of measurement used to build the Euclidean basis). And $|\det_{\vec{a}}(\vec{u}, \vec{v}, \vec{w})|$ is the positive volume (or volume) of the parallelepiped.

Exercise 1.5 Let $(\vec{b}_i)$ be defined by $\vec{b}_1 = -\vec{a}_1$, $\vec{b}_2 = \vec{a}_2$, $\vec{b}_3 = \vec{a}_3$ (change of orientation, drawing). Prove: $\det_{\vec{b}} = -\det_{\vec{a}}$ (the definition of a determinant is basis dependent).

Answer. $\det_{\vec{a}}$ and $\det_{\vec{b}}$ being tri-linear alternated, $\det_{\vec{b}}(\vec{a}_1, \vec{a}_2, \vec{a}_3) = \det_{\vec{b}}(-\vec{b}_1, \vec{b}_2, \vec{b}_3) = -\det_{\vec{b}}(\vec{b}_1, \vec{b}_2, \vec{b}_3) \overset{(1.6)}{=} -1 \overset{(1.6)}{=} -\det_{\vec{a}}(\vec{a}_1, \vec{a}_2, \vec{a}_3)$, thus (tri-linearity) $\det_{\vec{b}}(\vec{u}, \vec{v}, \vec{w}) = -\det_{\vec{a}}(\vec{u}, \vec{v}, \vec{w})$ for all $\vec{u}, \vec{v}, \vec{w}$. ▪

1.4 Orientation of a basis

Two bases $(\vec{a}_i)$ and $(\vec{b}_i)$ have the same orientation iff $\det_{\vec{a}}(\vec{b}_1, \vec{b}_2, \vec{b}_3) > 0$ (i.e. iff $\det_{\vec{b}}(\vec{a}_1, \vec{a}_2, \vec{a}_3) > 0$). We then also say that the basis $(\vec{b}_i)$ is positively oriented relatively to the basis $(\vec{a}_i)$.

(Euclidean setting: Right-handed bases are usually declared to be positively oriented.)

In other words, with $P = [P_{ij}]$ the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$, i.e. with $[\vec{b}_j]_{\vec{e}} = \begin{pmatrix} P_{1j} \\ P_{2j} \\ P_{3j} \end{pmatrix}$ (the j-th column of P) for all j: Two bases $(\vec{a}_i)$ and $(\vec{b}_i)$ have the same orientation iff $\det_M(P) > 0$.

1.5 Vector products and determinants

(1.3) and (1.7) immediately give: For all $\vec{u},\vec{v},\vec{w}\in V$,

$$\vec{u}\bullet_a(\vec{v}\times_a\vec{w})=\det_{\vec{a}}(\vec{u},\vec{v},\vec{w}). \tag{1.9}$$

Proposition 1.6 *Let $(\vec{b}_i)$ be another $(\cdot,\cdot)_a$-orthonormal basis (i.e. $\vec{b}_i\bullet_a\vec{b}_j=\delta_{ij}$ for all i,j, i.e. $(\cdot,\cdot)_b=(\cdot,\cdot)_a$). Then*

$$\begin{cases}\det_{\vec{b}}=\pm\det_{\vec{a}}, & i.e.\quad \det_{\vec{b}}(\vec{a}_1,\vec{a}_2,\vec{a}_3)=\pm 1,\\ \times_a=\pm\times_b, & i.e.\quad \vec{v}\times_b\vec{w}=\pm\vec{v}\times_a\vec{w},\ \ \forall\vec{v},\vec{w}\in V,\end{cases} \tag{1.10}$$

with + iff the bases have the same orientation.

Proof. Let $P:=\big([\vec{b}_1]_{\vec{a}}\quad[\vec{b}_2]_{\vec{a}}\quad[\vec{b}_3]_{\vec{a}}\big)$ (the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$). We get

$$\begin{aligned}\det_{\vec{a}}(\vec{b}_1,\vec{b}_2,\vec{b}_3)&=\det_M([\vec{b}_1]_{\vec{a}},[\vec{b}_2]_{\vec{a}},[\vec{b}_3]_{\vec{a}})=\det_M(P.[\vec{a}_1]_{\vec{a}},P.[\vec{a}_2]_{\vec{a}},P.[\vec{a}_3]_{\vec{a}})\\&=\det_M(P)\det_M([\vec{a}_1]_{\vec{a}},[\vec{a}_2]_{\vec{a}},[\vec{a}_3]_{\vec{a}})=\det_M(P)\det_M(I)=\det_M(P)\\&=\det_M(P)\det_{\vec{b}}(\vec{b}_1,\vec{b}_2,\vec{b}_3).\end{aligned} \tag{1.11}$$

And $\det_M(P)=\pm 1$ with + iff the bases have the same orientation. Thus (1.10).

Thus $(\vec{v}\times_b\vec{w})\bullet_b\vec{z}=\det_{\vec{b}}(\vec{v},\vec{w},\vec{z})=\pm\det_{\vec{a}}(\vec{v},\vec{w},\vec{z})=\pm(\vec{v}\times_a\vec{w})\bullet_a\vec{z}=\pm(\vec{v}\times_a\vec{w})\bullet_b\vec{z}$ for all $\vec{z}$ (since $(\cdot,\cdot)_b=(\cdot,\cdot)_a$), thus $\vec{v}\times_b\vec{w}=\pm\vec{v}\times_a\vec{w}$, with + iff the bases have the same orientation. Thus (1.10). ▪

Exercise 1.7 Let $(\vec{e}_i)$ be a basis in V. Prove that $\vec{v}\times_e\vec{w}$ is a "contravariant vector", i.e. satisfies the change of basis formula $[\vec{v}\times_e\vec{w}]_{\text{new}}=P^{-1}.[\vec{v}\times_e\vec{w}]_{\text{old}}$.

Answer. First proof: By definition a vector in V is called a contravariant vector, and $\vec{v}\times_e\vec{w}$ is in V, thus $\vec{v}\times_e\vec{w}$ is contravariant.

Second proof (verification of the change of basis formla): Let $(\vec{a}_i)$ be the old basis and $(\vec{b}_i)$ be the new basis, and let $P=P_{ij}$ be the transition matrix from $(\vec{a}_i)$ to $(\vec{b}_i)$, i.e. defined by $\vec{b}_j=\sum_i P_{ij}\vec{a}_i$ for all j. And let $g(\cdot,\cdot)$ be a scalar dot product (a symmetric definite positive bilinear form on V). The change of basis formulas give $[\vec{v}]_{\vec{b}}=P^{-1}.[\vec{v}]_{\vec{a}}$ for all $\vec{v}\in V$, and $[g]_{\vec{b}}=P^T.[g]_{\vec{a}}.P$. And, for all $\vec{u},\vec{v},\vec{w}\in V$, on the one hand $g(\vec{u},\vec{v}\times_e\vec{w})=[\vec{u}]_{\vec{a}}^T.[g]_{\vec{a}}.[\vec{v}\times_e\vec{w}]_{\vec{a}}$, and on the other hand

$$g(\vec{u},\vec{v}\times_e\vec{w})=[\vec{u}]_{\vec{b}}^T.[g]_{\vec{b}}.[\vec{v}\times_e\vec{w}]_{\vec{b}}=([\vec{u}]_{\vec{a}}^T.P^{-T}).(P^T.[g]_{\vec{a}}.P).[\vec{v}\times_e\vec{w}]_{\vec{b}}=[\vec{u}]_{\vec{a}}^T.[g]_{\vec{a}}.P.[\vec{v}\times_e\vec{w}]_{\vec{b}},$$

hence $[\vec{v}\times_e\vec{w}]_{\vec{a}}=P.[\vec{v}\times_e\vec{w}]_{\vec{b}}$, i.e. $[\vec{v}\times_e\vec{w}]_{\vec{b}}=P^{-1}.[\vec{v}\times_e\vec{w}]_{\vec{a}}$: It is the change of basis formula for vectors.

Third proof: $\vec{v}\times_e\vec{w}$ can also be defined with the Riesz representation theorem: The $(\cdot,\cdot)_e$-Riesz representation vector of the linear form $\ell_{e\vec{v}\vec{w}}:\vec{z}\to\ell_{e\vec{v}\vec{w}}(\vec{z})=\det_{\vec{e}}(\vec{v},\vec{w},\vec{z})$ is the vector $\vec{\ell}_{e\vec{v}\vec{w}}$ defined by $\ell_{e\vec{v}\vec{w}}.\vec{z}=(\vec{\ell}_{e\vec{v}\vec{w}},\vec{z})_e$ for all $\vec{z}\in V$, and $\vec{\ell}_{e\vec{v}\vec{w}}\overset{\text{noted}}{=}\vec{v}\times_e\vec{w}$ is contravariant since a Riesz-representation vector is. ▪

Exercise 1.8 Recall: Two distinct Euclidean products $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ satisfy $(\cdot,\cdot)_a=\lambda^2(\cdot,\cdot)_b$ where λ is the ratio of the the unit of measures (e.g. if $(\cdot,\cdot)_a$ is built with a metre and $(\cdot,\cdot)_b$ is built with a foot then $\lambda=0.3048$). Prove : If $(\vec{a}_i)$ is a $(\cdot,\cdot)_a$-orthonormal basis and $(\vec{b}_i)$ is a $(\cdot,\cdot)_b$-orthonormal basis, then $\det_{\vec{a}}=\pm\lambda^3\det_{\vec{b}}$ and $\times_a=\pm\lambda\times_b$, with + iff the bases have the same orientation.

Answer. With the transition matrix P from $(\vec{a}_i)$ to $(\vec{b}_i)$, i.e. $P=\big([\vec{b}_1]_{\vec{a}}\quad[\vec{b}_2]_{\vec{a}}\quad[\vec{b}_3]_{\vec{a}}\big)$, we get $\det_M(P)=\det_{\vec{a}}(\vec{b}_1,\vec{b}_2,\vec{b}_3)=\pm\lambda^3$ (ratio of algebraic volumes) $=\pm\lambda^3\det_{\vec{b}}(\vec{b}_1,\vec{b}_2,\vec{b}_3)$, thus $\det_{\vec{a}}=\pm\lambda^3\det_{\vec{b}}$, with + iff the bases have the same orientation. Thus $(\vec{v}\times_a\vec{w},\vec{z})_a=\det_{\vec{a}}(\vec{v},\vec{w},\vec{z})=\pm\lambda^3\det_{\vec{b}}(\vec{v},\vec{w},\vec{z})=\pm\lambda^3(\vec{v}\times_b\vec{w},\vec{z})_b=\pm\lambda^3\frac{1}{\lambda^2}(\vec{v}\times_b\vec{w},\vec{z})_b=\pm\lambda(\vec{v}\times_b\vec{w},\vec{z})_b$, for all $\vec{v},\vec{w},\vec{z}\in V$, thus $\vec{v}\times_a\vec{w}=\pm\lambda\vec{v}\times_b\vec{w}$, for all $\vec{v},\vec{w}\in V$. ▪

1.6 Basic properties

Proposition 1.9 *Let $(\vec{a}_i)$ be a basis in V and $\vec{v},\vec{w}\in V$. Then:*

1- If $\vec{v}\times_a\vec{z}=\vec{w}\times_a\vec{z}$ for all $\vec{z}$, then $\vec{v}=\vec{w}$.

2- With $(\cdot,\cdot)_a$ the associated dot product, $\vec{v}\times_a\vec{w}$ is $(\cdot,\cdot)_a$-orthogonal to $\vec{v}$ and to $\vec{w}$.

3- If $\vec{v}$ is parallel to $\vec{w}$ then $\vec{v}\times_a\vec{w}=0$. 3'- If $\vec{v}\times_a\vec{w}\neq 0$ then $\vec{v}$ is not parallel to $\vec{w}$.

4- If $\vec{v}$ is not parallel to $\vec{w}$ then $\vec{v}\times_a\vec{w}\neq 0$. 4'- If $\vec{v}\times_a\vec{w}=0$ then $\vec{v}$ is parallel to $\vec{w}$.

5- If $\vec{v}$ is not parallel to $\vec{w}$ then $(\vec{v},\vec{w},\vec{v}\times_a\vec{w})$ has the same orientation than $(\vec{a}_i)$.

6- Euclidean case:$||\vec{v}\times_a\vec{w}||_a$ is the area or the parallelogram $(\vec{v},\vec{w})$ (in the unit chosen for $(\vec{a}_i)$).

Proof. 1- $\vec{v}=\sum_i v_i\vec{a}_i$ and (1.3) give $[\vec{v}\times_a\vec{a}_1]_{\vec{a}}=\begin{pmatrix}0\\v_3\\-v_2\end{pmatrix}$, similarly $[\vec{w}\times_a\vec{a}_1]_{\vec{a}}=\begin{pmatrix}0\\w_3\\-w_2\end{pmatrix}$, thus $v_3=w_3$ and $v_2=w_2$. Similarly with $\vec{a}_2$ wich gives $v_1=w_1$.

2- With $\det_{\vec{a}}$ the associated determinant, $(\vec{v}\times_a\vec{w})\bullet_a\vec{v}=\det_{\vec{a}}(\vec{v},\vec{w},\vec{v})=0$ since $\det_{\vec{a}}$ is alternated, similarly $(\vec{v}\times_a\vec{w})\bullet_a\vec{w}=0$.

3- Trivial with (1.3). 3'- Contraposition.

4- If $\vec{v}$ is not parallel to $\vec{w}$ then let $\vec{z}\in V$ s.t. $(\vec{v},\vec{w},\vec{z})$ is a basis; Hence, $\det_{\vec{a}}(\vec{v},\vec{w},\vec{z})\neq 0$, thus $(\vec{v}\times_a\vec{w})\bullet_a\vec{z}\neq 0$, thus $\vec{v}\times_a\vec{w}\neq\vec{0}$. 4'- Contraposition.

5- $\det_{\vec{a}}(\vec{v},\vec{w},\vec{v}\times_a\vec{w})=^{(1.9)}(\vec{v}\times_a\vec{w})\bullet_a(\vec{v}\times_a\vec{w})=||\vec{v}\times_a\vec{w}||^2>0$ since $\vec{v}\not\parallel\vec{w}$.

6- If $\vec{v}$ is parallel to $\vec{w}$ then it is trivial (zero area). Otherwise $\vec{v}\times_a\vec{w}\neq\vec{0}$ thus $0\neq\det_{\vec{a}}(\vec{v},\vec{w},\frac{\vec{v}\times_a\vec{w}}{||\vec{v}\times_a\vec{w}||_a})=(\vec{v}\times_a\vec{w})\bullet_a\frac{\vec{v}\times_a\vec{w}}{||\vec{v}\times_a\vec{w}||_a}=||\vec{v}\times_a\vec{w}||_a=$ volume of the parallelepiped $(\vec{v},\vec{w},\frac{\vec{v}\times_a\vec{w}}{||\vec{v}\times_a\vec{w}||_a})$ (height 1). ▪

2 Antisymmetric endomorphism and the representation vectors

2.1 Dimension issue and Euclidean setting

Let $\mathcal{L}(V_1,V_2)$ be the space of linear maps from a vector space V_1 to a vector space V_2.

Continuing § 1.1. We will have to deal with linear maps $L\in\mathcal{L}(\overrightarrow{\mathbb{R}^3},V)$ where $\overrightarrow{\mathbb{R}^3}$ is the space of bi-point vectors (dimension [length]), and V is the space of velocities (dimension [length]/[time]) or the space of moments (dimension [force]$\times$ [length] = [masse]$\times$[length]^2/[time]^2). Hence the linear map L is not an endomorphism (although $\dim(V)=3$).

However, for quantification, we will use a (Euclidean) basis $(\vec{a}_i)\in\overrightarrow{\mathbb{R}^3}$ and a basis $(\vec{b}_i)\in V$ which use the same unit of length (thus $L.\vec{u}$ will give meaningful results); And we will abusively write $L\in\mathcal{L}(\overrightarrow{\mathbb{R}^3};\overrightarrow{\mathbb{R}^3})$, instead of $L\in\mathcal{L}(\overrightarrow{\mathbb{R}^3};V)$, and we will abusively say that L is an endomorphism.

2.2 Transpose of an endomorphism

Let V be a real vector space and $\mathcal{L}(V;V)$ be the set of endomorphisms $V\to V$ (it is a real vector space with the usual internal addition and external multiplication). Usual notation for a linear map: $L(\vec{v})=^{\text{noted}}L.\vec{v}$, hence $L.(\vec{v}+\lambda\vec{w})=L.\vec{v}+\lambda L.\vec{w}$ (distributivity notation = linearity).

Definition 2.1 Let $L\in\mathcal{L}(V;V)$ and let $(\cdot,\cdot)_g=.\bullet_g.$ be a scalar dot product in V. The transposed of L relative to $(\cdot,\cdot)_g$ is the linear map $L^T_g\in\mathcal{L}(V;V)$ defined by, for all $\vec{v},\vec{w}\in V$,

$$(L^T_g.\vec{w},\vec{v})_g=(\vec{w},L.\vec{v})_g\quad(=(L.\vec{v},\vec{w})_g),\tag{2.1}$$

i.e. $(L^T_g.\vec{w})\bullet_g\vec{v}=\vec{w}\bullet_g(L.\vec{v})$ $(=(L.\vec{v})\bullet_g\vec{w})$.

Matrix representation: Let $(\vec{a}_i)$ be a basis in V. Let $[L]_{\vec{a}}=[L_{ij}]$ be the matrix of L relative to $(\vec{a}_i)$, and let $[L^T_g]_{\vec{a}}=[(L^T_g)_{ij}]=$ the matrix of L^T_g relative to $(\vec{a}_i)$, i.e., for all j,

$$L.\vec{a}_j=\sum_{i=1}^3 L_{ij}\vec{a}_i\quad\text{and}\quad L^T_g.\vec{a}_j=\sum_i(L^T_g)_{ij}\vec{a}_i.\tag{2.2}$$

Let $[g]_{\vec{a}}$ be the matrix of $(\cdot,\cdot)_g$ relative to $(\vec{a}_i)$, i.e. $[g]_{\vec{a}}=[(\vec{a}_i,\vec{a}_j)_g]$. (2.1) gives $[\vec{v}]^T_{\vec{a}}.[g]_{\vec{a}}.[L^T_g]_{\vec{a}}.[\vec{w}]_{\vec{b}}=[\vec{v}]^T_{\vec{a}}.[L]^T_{\vec{a}}.[g]_{\vec{a}}.[\vec{w}]_{\vec{a}}$ for all $\vec{v},\vec{w}$, thus $[g]_{\vec{a}}.[L^T_g]_{\vec{a}}=[L]^T_{\vec{a}}.[g]_{\vec{a}}$, i.e.

$$[L^T_g]_{\vec{a}}=[g]^{-1}_{\vec{a}}.[L]^T_{\vec{a}}.[g]_{\vec{a}}.\tag{2.3}$$

In particular, if $(\vec{a}_i)$ is a $(\cdot,\cdot)_g$ orthonormal basis, then $[g]_{\vec{a}}=I$, thus $[L^T_g]_{\vec{a}}=[L]^T_{\vec{a}}$.

Euclidean setting: Useful result: If $(\cdot,\cdot)_a$ and $(\cdot,\cdot)_b$ are two Euclidean dot products (e.g. $(\cdot,\cdot)_a$ built with a metre and $(\cdot,\cdot)_b$ with a foot) then

$$L^T_a=L^T_b\overset{\text{noted}}{=}L^T\quad\text{(Euclidean setting)},\tag{2.4}$$

Indeed, $\exists\lambda>0$ s.t. $(\cdot,\cdot)_a=\lambda^2(\cdot,\cdot)_b$, thus $(L^T_a.\vec{w},\vec{v})_a=(\vec{w},L.\vec{v})_a=\lambda^2(\vec{w},L.\vec{v})_b=\lambda^2(L^T_b.\vec{w},\vec{v})_b=(L^T_b.\vec{w},\vec{v})_a$ for all $\vec{v},\vec{w}\in\overrightarrow{\mathbb{R}^3}$. So the transposed of an endomorphism in $\overrightarrow{\mathbb{R}^3}$ Euclidean doesn't depend on the unit of measurement (metre, foot,...) used to build a Euclidean basis.

Matrix representation: With $(\vec{e}_i)$ is a $(\cdot,\cdot)_e$-Euclidean basis in $V = \overrightarrow{\mathbb{R}^3}$ (so $[e]_{\vec{e}} = I$):

$$[L^T]_{\vec{e}} = [L]_{\vec{e}}^T, \quad \text{i.e.} \quad (L^T)_{ij} = L_{ji} \;\; \forall i,j. \tag{2.5}$$

Exercise 2.2 Check: The transposed map ${}^T_g : L \in \mathcal{L}(V;V) \to {}^T_g(L) := L_g^T \in \mathcal{L}(V;V)$ is linear.

Answer. $((L+\lambda M)_g^T.\vec{w},\vec{v})_g = (\vec{w},(L+\lambda M).\vec{v})_g = (\vec{w},L.\vec{v})_g + \lambda(\vec{w},M.\vec{v})_g = (L_g^T.\vec{w},\vec{v})_g + \lambda(M_g^T.\vec{w},\vec{v})_g = ((L_g^T+\lambda M_g^T).\vec{w},\vec{v})_g$, for all $\vec{v},\vec{w} \in V$ and $\lambda \in \mathbb{R}$, gives $(L+\lambda M)_g^T = L_g^T + \lambda M_g^T$. ∎

2.3 Symmetric and antisymmetric endomorphisms

Definition 2.3 Let $L \in \mathcal{L}(V;V)$ and let $(\cdot,\cdot)_g$ be a scalar dot product in V.

- L is $(\cdot,\cdot)_g$-symmetric iff $L_g^T = L$, i.e. $(L.\vec{w},\vec{v})_g = (\vec{w},L.\vec{v})_g$, $\forall \vec{v},\vec{w}$,
- L is $(\cdot,\cdot)_g$-antisymmetric iff $L_g^T = -L$, i.e. $(L.\vec{w},\vec{v})_g = -(\vec{w},L.\vec{v})_g$, $\forall \vec{v},\vec{w}$. (2.6)

Proposition 2.4 *The space of $(\cdot,\cdot)_g$-symmetric endomorphisms is a vector space. The space of $(\cdot,\cdot)_g$-antisymmetric endomorphisms is a vector space.*

Proof. $(L+\lambda M)_g^T = L_g^T + \lambda M_g^T = (\pm L) + \lambda(\pm M) = \pm(L+\lambda M)$ with $+$ iff L and M are $(\cdot,\cdot)_g$-symmetric and $-$ iff L and M are antisymmetric. Thus, vector sub-spaces of $\mathcal{L}(V;V)$. ∎

Euclidean setting: $(\cdot,\cdot)_g$ is a Euclidean dot product; With (2.4),

- L is Euclidean-symmetric iff $L^T = L$,
- L is Euclidean-antisymmetric iff $L^T = -L$. (2.7)

Hence if $(\vec{e}_i)$ is any euclidean basis (so $[g]_{\vec{e}} = \lambda^2 I$), (2.3) gives

- L is Euclidean-symmetric iff $[L^T]_{\vec{e}} = [L]_{\vec{e}}$,
- L is Euclidean-antisymmetric iff $[L^T]_{\vec{e}} = -[L]_{\vec{e}}$. (2.8)

2.4 Antisymmetric endomorphism and the representation vectors

Let $L \in \mathcal{L}(\overrightarrow{\mathbb{R}^3};\overrightarrow{\mathbb{R}^3})$. Let $(\vec{e}_i)$ be a Euclidean basis and $(\cdot,\cdot)_e$ be the associated Euclidean dot product. Suppose L is $(\cdot,\cdot)_e$-antisymmetric: $(2.6)_2$ gives $(L.\vec{e}_i,\vec{e}_i)_e = 0$, thus $L_{ii} = 0$ for all i, thus $L.\vec{e}_1 = c\vec{e}_2 - b\vec{e}_3$, $L.\vec{e}_2 = -c\vec{e}_1 + a\vec{e}_3$ and $L.\vec{e}_3 = b\vec{e}_1 - a\vec{e}_2$ for some $a,b,c \in \mathbb{R}$. And the vector $\vec{\omega}_e := a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3 \in \overrightarrow{\mathbb{R}^3}$ immediately satisfies

$$L.\vec{v} = \vec{\omega}_e \times_e \vec{v}. \tag{2.9}$$

I.e.

$$[L]_{\vec{e}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \quad \text{and} \quad [\vec{\omega}_e]_{\vec{e}} := \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{give} \quad L.\vec{v} = \vec{\omega}_e \times_e \vec{v}, \;\; \forall \vec{v} \in \overrightarrow{\mathbb{R}^3}. \tag{2.10}$$

NB: The representation vector $\vec{\omega}_e$ (of L) is ***not*** intrinsic to L, because it depends on the choice of a basis by an observer. See next exercise 2.6.

Definition 2.5 $\vec{\omega}_e = a\vec{e}_1 + b\vec{e}_2 + c\vec{e}_3$ is called the representation vector of the antisymmetric endomorphism L relative to the Euclidean basis $(\vec{e}_i)$.

Exercise 2.6 Let $(\vec{z}_i)$ be another $(\cdot,\cdot)_e$-orthonormal basis; Prove: $\vec{\omega}_z = \pm\vec{\omega}_e$ with $+$ iff the bases have the same orientation. Let $(\vec{b}_i)$ be a basis s.t. $(\vec{e}_i,\vec{e}_j)_e = \lambda^2(\vec{b}_i,\vec{b}_j)_e$ for some $\lambda > 0$ (e.g. $(\vec{e}_i)$ is a Euclidean basis made with the metre and $(\vec{b}_i)$ is a Euclidean basis made with the foot); Prove: $\vec{\omega}_b = \pm\lambda\vec{\omega}_e$ with $+$ iff the bases have the same orientation.

Answer. (1.10) gives $\times_z = \pm\times_e$, thus $\vec{\omega}_e \times_e \vec{v} = L.\vec{v} = \vec{\omega}_z \times_z \vec{v} = \pm\vec{\omega}_z \times_e \vec{v}$, for all $\vec{v}$, thus $\vec{\omega}_z = \pm\vec{\omega}_e$, with $+$ iff the bases have the same orientation. For $(\vec{b}_i)$, apply exercise 1.8. ∎

2.5 Interpretation ($\pi/2$ rotation and dilation)

Consider (2.9) (or (2.10)), and let $\omega_e = ||\vec{\omega}_e||_e = \sqrt{a^2+b^2+c^2}$. Then define the positively oriented Euclidean basis $(\vec{b}_i)$ s.t. $\vec{b}_3 := \frac{\vec{\omega}_e}{\omega_e}$. We get (direct calculation see exercise 2.7)

$$[L]_{\vec{b}} = \omega_e \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \omega_e \begin{pmatrix} 0 & -\sin(\frac{\pi}{2}) & 0 \\ \sin(\frac{\pi}{2}) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad [\vec{\omega}_e]_{\vec{b}} = \omega_e \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.11}$$

So $L.\vec{v}$ rotates a vector $\vec{v} = v_1\vec{b}_1 + v_2\vec{b}_2 \in \mathrm{Vect}\{\vec{b}_1, \vec{b}_2\}$ through an angle $\frac{\pi}{2}$ radians in the plane $\mathrm{Vect}\{\vec{b}_1, \vec{b}_2\}$ and dilates by a factor ω_e (in particular $L.\vec{b}_1 = \omega_e\vec{b}_2$ and $L.\vec{b}_2 = -\omega_e\vec{b}_1$). And $L.\vec{b}_3 = \vec{0}$.

Exercise 2.7 Give a Euclidean basis $(\vec{b}_i)$ s.t. $[L]_{\vec{e}}$ is given by (2.11).

Answer. $[\vec{b}_3]_{\vec{e}} = \frac{1}{\omega_e}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$; Let $[\vec{b}_1]_{|\vec{e}} = \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix} -b \\ a \\ 0 \end{pmatrix}$, so $\vec{b}_1$ is a unit vector $\perp \vec{b}_3$. Then choose $\vec{b}_2 = \vec{b}_3 \times_e \vec{b}_1$, i.e. $[\vec{b}_2]_{|\vec{e}} = \frac{1}{\sqrt{a^2+b^2}}\frac{1}{\omega_e}\begin{pmatrix} -ac \\ -bc \\ a^2+b^2 \end{pmatrix}$. Thus $(\vec{b}_i)$ is a direct orthonormal basis. With $P = \begin{pmatrix} [\vec{b}_1]_{|\vec{e}} & [\vec{b}_2]_{|\vec{e}} & [\vec{b}_3]_{|\vec{e}} \end{pmatrix}$ the transition matrix from $(\vec{e}_i)$ to $(\vec{b}_i)$ we have $[L]_{|\vec{b}} = P^{-1}.[L]_{|\vec{e}}.P$ (change of basis formula for endomorphisms), with $P^{-1} = P^T$ (change of orthonormal basis): We get (2.11). ∴

3 Antisymmetric matrix and the pseudo-vector representation

3.1 The pseudo-vector product

Let $\mathcal{M}_{31} = \{\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} : v_1, v_2, v_3 \in \mathbb{R}\}$ be the set of real $3*1$ column matrices. Let $\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} =^{\text{noted}} [\vec{v}]$.

Definition 3.1 A column matrix $[\vec{v}] \in \mathcal{M}_{31}$ is also called a pseudo-vector.

Definition 3.2 The pseudo-vector product is the function $\overset{\circlearrowleft}{\times} : \mathcal{M}_{31} \times \mathcal{M}_{31} \to \mathcal{M}_{31}$ defined by

$$\overset{\circlearrowleft}{\times}([\vec{v}], [\vec{w}]) = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix} \overset{\text{noted}}{=} [\vec{v}] \overset{\circlearrowleft}{\times} [\vec{w}, \quad \text{when} \quad [\vec{v}] = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \quad \text{and} \quad [\vec{w}] = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}, \tag{3.1}$$

and the column matrix $[\vec{v}] \overset{\circlearrowleft}{\times} [\vec{w}]$ is called the pseudo-vector product of $[\vec{v}]$ and $[\vec{w}]$.

Remark 3.3 $\mathcal{M}_{31}$ being a vector space, the pseudo-vector product $\overset{\circlearrowleft}{\times}$ can also be defined to be the vector product $\overset{\circlearrowleft}{\times} := \times_E$ in $\mathcal{M}_{31}$, where $([\vec{E}_i])$ is the canonical basis in $\mathcal{M}_{31}$, i.e. $[\vec{E}_1] := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $[\vec{E}_2] := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $[\vec{E}_3] := \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ (three column matrices), the unit 1 being the neutral element of the multiplication in $\mathbb{R}$. This unit 1 is not linked to any "unit of measurement": It is theoretical. And a "Euclidean basis" is meaningless in $\mathcal{M}_{31}$. ∴

3.2 Antisymmetric matrix and the pseudo-vector representation

Let $M \in \mathcal{M}_{33}$ be an antisymmetric matrix, so there exists $a, b, c \in \mathbb{R}$ s.t.

$$M = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}, \quad \text{and let} \quad \overset{\circlearrowleft}{\omega} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \tag{3.2}$$

We immediately get, for all $[\vec{v}] \in \mathcal{M}_{31}$,

$$\boxed{M.[\vec{v}] = \overset{\circlearrowleft}{\omega} \overset{\circlearrowleft}{\times} [\vec{v}]}. \tag{3.3}$$

The column matrix (the pseudo-vector) $\overset{\circlearrowleft}{\omega} \in \mathcal{M}_{31}$ is called the pseudo-vector representation (column matrix representation) of the matrix M.

3.3 Pseudo-representation vectors of an antisymmetric endomorphism

Let $(\vec{e}_i)$ be a Euclidean basis and $(\cdot,\cdot)_e$ its associated Euclidean dot product. Let $L \in \mathcal{L}(\vec{\mathbb{R}^3};\vec{\mathbb{R}^3})$ be $(\cdot,\cdot)_e$-antisymmetric. Then (2.9) immediately gives, for all $[\vec{v}] \in \mathcal{M}_{31}$,

$$[L]_{\vec{e}}.[\vec{v}] = \overset{\circlearrowleft}{\omega} \overset{\circlearrowleft}{\times} [\vec{v}] \quad \text{where} \quad \overset{\circlearrowleft}{\omega} := [\vec{\omega}_e]_{\vec{e}}, \tag{3.4}$$

and where $\overset{\circlearrowleft}{\omega}$ should be written $\overset{\circlearrowleft}{\omega}_e$ since it depends on the basis $(\vec{e}_i)$ used to define $[L]_{\vec{e}}$ (depends on the chosen unit of measurement used to build the Euclidean basis $(\vec{e}_i)$). The matrix $\overset{\circlearrowleft}{\omega}$ is called the pseudo-representation vector of L relative to $(\vec{e}_i)$.

4 Screw (torsor)

4.0 Reminder

• Let Dom be an open set in $\mathbb{R}^3$ and let $\vec{u} : \left\{ \begin{array}{ll} Dom & \to \vec{\mathbb{R}^3} \\ A & \to \vec{u}(A) \end{array} \right\}$ (so Dom is the domain of definition of $\vec{u}$ e.g. the position in space of some material at some time t). The function $\vec{u}$ is differentiable at $A \in Dom$ iff there exists $L_{\vec{u}}(A) \in \mathcal{L}(\vec{\mathbb{R}^3};\vec{\mathbb{R}^3})$ (linear) s.t. $\vec{u}(B) = \vec{u}(A) + L_{\vec{u}}(A).\overrightarrow{AB} + o(||\overrightarrow{AB}||)$ near A (first order Taylor expansion). In which case $L_{\vec{u}}(A) =^{\text{noted}} d\vec{u}(A)$, called the differential of $\vec{u}$ at A.

• $\vec{u} : Dom \to \vec{\mathbb{R}^3}$ is affine iff there exists $L_{\vec{u}} \in \mathcal{L}(\vec{\mathbb{R}^3};\vec{\mathbb{R}^3})$ s.t., for all $A \in Dom$ and B near A,

$$\vec{u}(B) = \vec{u}(A) + L_{\vec{u}}.\overrightarrow{AB}. \tag{4.1}$$

$L_{\vec{u}}$ is called "the associated linear map" with $\vec{u}$. Thus $\vec{u}$ is differentiable in Dom, and, at any A, its differential $L_{\vec{u}}(A) = d\vec{u}(A) =^{\text{noted}} d\vec{u} =^{\text{noted}} L_{\vec{u}}$ is independent of A. (Here the first order Taylor expansion reads $\vec{u}(B) = \vec{u}(A) + d\vec{u}.\overrightarrow{AB} + o(||\overrightarrow{AB}||)$ with $o(||\overrightarrow{AB}||) = 0$.)

• A vector field in $\mathbb{R}^3$ is a function $\widetilde{\vec{u}} : \left\{ \begin{array}{ll} Dom & \to Dom \times \vec{\mathbb{R}^3} \\ A & \to \widetilde{\vec{u}}(A) := (A, \vec{u}(A)) \end{array} \right\}$, the couple $\widetilde{\vec{u}}(A) := (A, \vec{u}(A))$ being a "pointed vector at A", or "a vector at A". Drawing: $\vec{u}(A)$ has to be drawn at A, nowhere else. To compare with a vector $\vec{v} \in \vec{\mathbb{R}^3}$ which can be drawn anywhere (also called a free vector).

The sum of two vector fields $\widetilde{\vec{u}}, \widetilde{\vec{w}} : Dom \to \vec{\mathbb{R}^3}$ and the multiplication by a scalar λ are defined by, at any $A \in Dom$,

$$\widetilde{\vec{u}}(A) + \widetilde{\vec{w}}(A) = (A, \vec{u}(A) + \vec{w}(A)), \quad \text{and} \quad \lambda\widetilde{\vec{u}}(A) = (A, \lambda\vec{u}(A)) \tag{4.2}$$

(usual rules for "vectors at A").

If there is no ambiguity then, to lighten the notations, $\widetilde{\vec{u}}(A) =^{\text{noted}} \vec{u}(A)$ (pointed vector).

The differential of a vector field $\widetilde{\vec{u}} : Dom \to Dom \times \vec{\mathbb{R}^3}$ at a point A is the "field of endomorphisms" $d\widetilde{\vec{u}} : Dom \to Dom \times \mathcal{L}(\vec{\mathbb{R}^3};\vec{\mathbb{R}^3})$ defined by $d\widetilde{\vec{u}}(A) = (A, d\vec{u}(A))$ (an endomorphism at A).

• An affine vector field $\widetilde{\vec{u}} : \left\{ \begin{array}{ll} Dom & \to Dom \times \vec{\mathbb{R}^3} \\ A & \to \widetilde{\vec{u}}(A) := (A, \vec{u}(A)) \end{array} \right\}$ is a vector field s.t. $\vec{u} : Dom \to \vec{\mathbb{R}^3}$ is affine.

4.1 Definition

Definition 4.1 *A screw (a torsor) is a Euclidean antisymmetric affine vector field, i.e. a function*

$$\widetilde{\vec{u}} : \left\{ \begin{array}{ll} Dom & \to Dom \times \vec{\mathbb{R}^3} \\ A & \to \widetilde{\vec{u}}(A) := (A, \vec{u}(A)) \end{array} \right\} \quad \textit{s.t.} \quad \vec{u} : Dom \times \vec{\mathbb{R}^3} \textit{ is affine antisymmetric.} \tag{4.3}$$

To lighten the notations $\widetilde{\vec{u}}(A) =^{noted} \vec{u}(A)$ *(pointed vector at A). So, for all $A, B \in Dom$:*

$$\boxed{\vec{u}(B) = \vec{u}(A) + L_{\vec{u}}.\overrightarrow{AB} \quad \textit{where} \quad {L_{\vec{u}}}^T = -L_{\vec{u}}}. \tag{4.4}$$

$\vec{u}(A)$ is called the moment of the screw $\vec{u}$ at A (or moment of the torsor $\vec{u}$ at A).

If $\vec{u} = \vec{0}$ then $\widetilde{\vec{u}}$ is a degenerate screw (a degenerate torsor).

Exercise 4.2 Let $\mathcal{S}$ be the set of the screws $\vec{u} : Dom \to \overrightarrow{\mathbb{R}^3}$. Prove: $\mathcal{S}$ is a vector space, and the map $\ell : \left\{ \begin{array}{ll} \mathcal{S} & \to \mathcal{L}(\overrightarrow{\mathbb{R}^3};\overrightarrow{\mathbb{R}^3}) \\ \vec{u} & \to \ell(\vec{u}) = L_{\vec{u}} \end{array} \right\}$ is linear.

Answer. If $\vec{u}_1, \vec{u}_2 \in \mathcal{S}$ and $\lambda \in \mathbb{R}$ then $\vec{u}_1{+}\lambda\vec{u}_2$ is affine antisymmetric: Indeed, at B, $(\vec{u}_1{+}\lambda\vec{u}_2)(B) = \vec{u}_1(B) + \lambda\vec{u}_2(B) = (\vec{u}_1(A){+}L_{\vec{u}_1}.\overrightarrow{AB}){+}\lambda(\vec{u}_2(A){+}L_{\vec{u}_2}.\overrightarrow{AB}) = (\vec{u}_1{+}\lambda\vec{u}_2)(A){+}(L_{\vec{u}_1}{+}\lambda L_{\vec{u}_2}).\overrightarrow{AB}$ with $L_{\vec{u}_1}{+}\lambda L_{\vec{u}_2}$ antisymmetric since $L_{\vec{u}_1}$ and $L_{\vec{u}_2}$ are; Thus $\vec{u}_1{+}\lambda\vec{u}_2$ is affine and $L_{\vec{u}_1+\lambda\vec{u}_2} = L_{\vec{u}_1}{+}\lambda L_{\vec{u}_2}$ is the associated linear function. Thus $\ell(\vec{u}_1{+}\lambda\vec{u}_2) = L_{\vec{u}_1+\lambda\vec{u}_2} = L_{\vec{u}_1}{+}\lambda L_{\vec{u}_2} = \ell(\vec{u}_1) + \lambda\ell(\vec{u}_2)$ (linearity). ▪

4.2 Constant screw

Definition 4.3 A constant screw $\vec{u}$ is a non degenerate screw ($\vec{u} \neq \vec{0}$) s.t.

$$\forall A, B \in Dom, \quad \vec{u}(A) = \vec{u}(B). \tag{4.5}$$

4.3 Euclidean setting: Resultant vector and resultant (pseudo-vector)

Euclidean setting: Euclidean basis $(\vec{e}_i)$ in $\overrightarrow{\mathbb{R}^3}$, associated Euclidean dot product $(\cdot,\cdot)_e$ (needed to define the transposed of a linear map), associated vector product $\times_e$ (needed to represent an antisymmetric endomorphism with a vector).

Consider a screw $\vec{u} : Dom \to \mathbb{R}^3$, given as in (4.4). With $[L_{\vec{u}}]_{\vec{e}} = \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$ and $[\vec{\omega}_e]_{\vec{e}} := \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \overset{\circlearrowleft}{\omega}$, cf. (2.10), we get, for all $A, B \in Dom$,

$$\vec{u}(B) = \vec{u}(A) + \vec{\omega}_e \times_e \overrightarrow{AB}, \quad \text{i.e.} \quad [\vec{u}(B)]_{\vec{e}} = [\vec{u}(A)]_{\vec{e}} + \overset{\circlearrowleft}{\omega} \overset{\circlearrowleft}{\times} [\overrightarrow{AB}]_{\vec{e}}. \tag{4.6}$$

Definition 4.4 The vector $\vec{\omega}_e \in \overrightarrow{\mathbb{R}^3}$ is the resultant vector of the screw $\vec{u}$ relative to $(\vec{e}_i)$.

The vector reduction elements at $A \in Dom$ are the vectors $\vec{\omega}_e \in \overrightarrow{\mathbb{R}^3}$ and $\vec{u}(A) \in \overrightarrow{\mathbb{R}^3}$, often written as the couple of vectors $(\vec{\omega}_e, \vec{u}(A)) =^{\text{noted}} \begin{pmatrix} \vec{\omega}_e \\ \vec{u}(A) \end{pmatrix}$ (relative to $(\vec{e}_i)$).

Definition 4.5 The pseudo-vector (the matrix) $\overset{\circlearrowleft}{\omega} := [\vec{\omega}_e]_{\vec{e}}$ is the resultant of the screw $\vec{u}$ relative to $(\vec{e}_i)$.

The reduction elements at $A \in Dom$ are the pseudo-vectors $\overset{\circlearrowleft}{\omega} := [\vec{\omega}_e]_{\vec{e}} \in \mathcal{M}_{31}$ and $[\vec{u}(A)]_{\vec{e}} \in \mathcal{M}_{31}$, often written as the couple of matrices $(\overset{\circlearrowleft}{\omega}, [\vec{u}(A)]_{\vec{e}}) =^{\text{noted}} \begin{pmatrix} \overset{\circlearrowleft}{\omega} \\ [\vec{u}(A)]_{\vec{e}} \end{pmatrix}$ (relative to $(\vec{e}_i)$).

NB: Recall: The representation vector $\vec{\omega}_e$ (of $L_{\vec{u}}$) is ***not*** intrinsic to $L_{\vec{u}}$, because it depends on the choice of a basis by an observer, cf. exercise 2.6. Thus the pseudo-vector $\overset{\circlearrowleft}{\omega}$ is ***not*** intrinsic to $L_{\vec{u}}$ either.

Remark 4.6 (4.6) is sometimes abusively written $\vec{u}(B) = \vec{u}(A) + \vec{\omega} \times \overrightarrow{AB}$ (no reference to any basis) which causes misunderstandings and confusions between vectors and pseudo-vectors (matrices). ▪

Exercise 4.7 Let $\vec{u}$ be a screw. For all $\lambda \in \mathbb{R}$ and $A, B \in \mathbb{R}^3$, prove:

$$\vec{u}(A + \lambda\vec{\omega}_e) = \vec{u}(A), \tag{4.7}$$

then

$$\vec{u}(B) \bullet_e \vec{\omega}_e = \vec{u}(A) \bullet_e \vec{\omega}_e, \quad \text{thus} = \text{constant, called the screw invariant,} \tag{4.8}$$

and $(\vec{u}(B) \bullet_e \frac{\vec{\omega}_e}{||\vec{\omega}_e||_e})\frac{\vec{\omega}_e}{||\vec{\omega}_e||_e}$ is the vector invariant, and

$$\vec{u}(B) \bullet_e \overrightarrow{AB} = \vec{u}(A) \bullet_e \overrightarrow{AB}, \quad \text{called the equi-projectivity property.} \tag{4.9}$$

Answer. $Z = A + \lambda\vec{\omega}_e$ gives $\overrightarrow{AZ} = \lambda\vec{\omega}_e$, thus $\vec{u}(Z) =^{(4.6)} \vec{u}(A) + \vec{\omega}_e \times_e (\lambda\vec{\omega}_e) = \vec{u}(A) + \vec{0}$, i.e. (4.7).

Then $\vec{u}(B) =^{(4.6)} \vec{u}(A) + \vec{\omega}_e \times_e \overrightarrow{AB}$ and $\vec{\omega}_e \times_e \overrightarrow{AB}$ orthogonal to $\vec{\omega}_e$ and $\overrightarrow{AB}$ give (4.8)-(4.9). ▪

Exercise 4.8 Choose a basis $(\vec{e}_i)$ (and $\mathcal{S}$ is the set of screws). Let A be fixed and define the function $f_A : (\vec{z},\vec{w}) \in \overrightarrow{\mathbb{R}^3} \times_e \overrightarrow{\mathbb{R}^3} \to \vec{u} = f_A(\vec{z},\vec{w}) \in \mathcal{S}$ by $f_A(\vec{z},\vec{w})(B) := \vec{z} + \vec{w} \times_e \overrightarrow{AB} = \vec{u}(B)$. Prove that f_A is linear and bijective (is one-to-one and onto).

Answer. Linearity: $f_A((\vec{z}_1,\vec{w}_1) + \lambda(\vec{z}_2,\vec{w}_2))(B) = f_A(\vec{z}_1+\lambda\vec{z}_2, \vec{w}_1+\lambda\vec{w}_2)(B) = \vec{z}_1+\lambda\vec{z}_2 + (\vec{w}_1+\lambda\vec{w}_2) \times_e \overrightarrow{AB} = \vec{z}_1+\vec{w}_1 \times_e \overrightarrow{AB} + \lambda(\vec{z}_2+\vec{w}_2 \times_e \overrightarrow{AB}) = (f_A(\vec{z}_1,\vec{w}_1) + \lambda f_A(\vec{z}_2,\vec{w}_2))(B)$.

One-to-one: $f_A(\vec{z},\vec{w}) = 0$ iff $\vec{z} + \vec{w} \times_e \overrightarrow{AB} = \vec{0}$ for all B, in particular $B = A$ gives $\vec{z} = \vec{0}$ and then $\vec{w} = \vec{0}$.

Onto: Let $\vec{u} \in \mathcal{S}$, $\vec{u}(B) = \vec{u}(A) + \vec{\omega}_e \times_e \overrightarrow{AB}$, and take $\vec{z} = \vec{u}(A)$ and $\vec{w} = \vec{\omega}_e$. ∎

Exercise 4.9 Choose a basis $(\vec{e}_i)$, write $\times_e{=}\times$, $\bullet_e = \bullet$, $\vec{\omega}_e{=}\vec{\omega}$. Let $\vec{u}_1, \vec{u}_2 \in \mathcal{S}$, $\vec{u}_1(B) = \vec{u}_1(A) + \vec{\omega}_1 \times \overrightarrow{AB}$ and $\vec{u}_2(B) = \vec{u}_2(A) + \vec{\omega}_2 \times \overrightarrow{AB}$. Define the screw $\langle \vec{u}_1, \vec{u}_2 \rangle$ by $\langle \vec{u}_1, \vec{u}_2 \rangle(A) = \vec{\omega}_1 \bullet \vec{u}_2(A) + \vec{\omega}_2 \bullet \vec{u}_1(A)$. Prove $\langle \vec{u}_1, \vec{u}_2 \rangle$ is constant.

Answer. $\vec{\omega}_1 \bullet \vec{u}_2(B) + \vec{\omega}_2 \bullet \vec{u}_1(B) = \vec{\omega}_1 \bullet (\vec{u}_2(A) + \vec{\omega}_2 \times \overrightarrow{AB}) + \vec{\omega}_2 \bullet (\vec{u}_1(A) + \vec{\omega}_1 \times \overrightarrow{AB}) = \vec{\omega}_1 \bullet \vec{u}_2(A) + \vec{\omega}_2 \bullet \vec{u}_1(A) + \vec{\omega}_1 \bullet (\vec{\omega}_2 \times \overrightarrow{AB}) + \vec{\omega}_2 \bullet (\vec{\omega}_1 \times \overrightarrow{AB})$, with $\vec{\omega}_1 \bullet (\vec{\omega}_2 \times \overrightarrow{AB}) + \vec{\omega}_2 \bullet (\vec{\omega}_1 \times \overrightarrow{AB}) = \det_{\vec{e}}(\vec{\omega}_1, \vec{\omega}_2, \overrightarrow{AB}) + \det_{\vec{e}}(\vec{\omega}_2, \vec{\omega}_1, \overrightarrow{AB})$ hence $= 0$, thus $\vec{\omega}_1 \bullet \vec{u}_2(B) + \vec{\omega}_2 \bullet \vec{u}_1(B) = \vec{\omega}_1 \bullet \vec{u}_2(A) + \vec{\omega}_2 \bullet \vec{u}_1(A)$, for all A, B. ∎

4.4 Central axis

Let $(\vec{e}_i)$ be a Euclidean basis. Let $\vec{u} : Dom \to \overrightarrow{\mathbb{R}^3}$ be a screw; $\vec{u}$ being affine, it can be extended to $\vec{u} : \mathbb{R}^3 \to \overrightarrow{\mathbb{R}^3}$, i.e. the domain of definition Dom of $\vec{u}$ is extended to the whole space $\mathbb{R}^3$ (the material is extended with "zero density" to the whole space). Let $L_{\vec{u}}$ be the associated antisymmetric endomorphism and let $\vec{\omega}_e \in \overrightarrow{\mathbb{R}^3}$ defined by $L_{\vec{u}}.() = \vec{\omega}_e \times_e ()$.

Definition 4.10 The central axis of a non constant screw is the set of central points defined by

$$\mathrm{Ax}(\vec{u}) = \{C \in \mathbb{R}^3 : \vec{u}(C) \parallel \vec{\omega}_e\}. \tag{4.10}$$

i.e. $\mathrm{Ax}(\vec{u}) = \{C \in \mathbb{R}^3 : \exists\lambda \in \mathbb{R},\ \vec{u}(C) = \lambda\vec{\omega}_e\}$.

Proposition 4.11 *Let $\vec{u}$ be a non constant screw. Let $O \in \mathbb{R}^3$. Define $C_0 := O + \frac{1}{||\vec{\omega}_e||^2}\vec{\omega}_e \times_e \vec{u}(O) \in \mathbb{R}^3$, i.e. C_0 is defined by*

$$\overrightarrow{OC_0} = \frac{1}{||\vec{\omega}_e||^2}\vec{\omega}_e \times_e \vec{u}(O). \tag{4.11}$$

1- $C_0 \in \mathrm{Ax}(\vec{u})$, and

$$\mathrm{Ax}(\vec{u}) = C_0 + \mathrm{Vect}\{\vec{\omega}_e\}. \tag{4.12}$$

2- $\vec{u}$ is constant in $\mathrm{Ax}(\vec{u})$.

3- $C \in \mathrm{Ax}(\vec{u})$ iff $C = \arg\min_{A\in\mathbb{R}^3} ||\vec{u}(A)||_e$ (i.e. iff $||\vec{u}(C)||_e = \min_{A\in\mathbb{R}^3} ||\vec{u}(A)||_e$).

3'- $||\vec{u}(B)||_e > ||\vec{u}(C)||_e$ for all $C \in \mathrm{Ax}(\vec{u})$ and all $B \notin \mathrm{Ax}(\vec{u})$.

4- For all $B \in \mathbb{R}^3$,

$$\vec{u}(B) = \vec{u}(C_0) + \vec{\omega}_e \times_e \overrightarrow{C_0B} \quad \in \mathrm{Vect}\{\vec{\omega}_e\} \oplus^\perp \mathrm{Vect}\{\vec{\omega}_e\}^\perp \quad \textit{(orthogonal sum)}. \tag{4.13}$$

Proof. 1- $\vec{u}(C_0) = \vec{u}(O) + \vec{\omega}_e \times_e \overrightarrow{OC_0} = \vec{u}(O) + \vec{\omega}_e \times_e (\frac{1}{||\vec{\omega}_e||^2}\vec{\omega}_e \times_e \vec{u}(O)) = \vec{u}(O) + \frac{1}{||\vec{\omega}_e||^2}(\vec{\omega}_e \bullet_e \vec{u}(O))\vec{\omega}_e - \frac{1}{||\vec{\omega}_e||^2}||\vec{\omega}_e||^2\vec{u}(O) = \frac{1}{||\vec{\omega}_e||^2}(\vec{\omega}_e \bullet_e \vec{u}(O))\vec{\omega}_e$ is parallel to $\vec{\omega}_e$, thus $C_0 \in \mathrm{Ax}(\vec{u})$.

Then $\vec{u}(C_0 + \lambda\vec{\omega}_e) = \vec{u}(C_0) + \vec{0}$ for all λ (because $\vec{\omega}_e \times_e \vec{\omega}_e = \vec{0}$), thus $\mathrm{Ax}(\vec{u}) \supset C_0 + \mathrm{Vect}\{\vec{\omega}_e\}$.

If $B \notin C_0{+}\mathrm{Vect}\{\vec{\omega}_e\}$, then $\overrightarrow{C_0B} \nparallel \vec{\omega}_e$, i.e. $\vec{\omega}_e {\times_e} \overrightarrow{C_0B} \neq \vec{0}$, thus $\vec{u}(B) = \vec{u}(C_0) {+} \vec{\omega}_e {\times_e} \overrightarrow{C_0B} \in \mathrm{Vect}\{\vec{\omega}_e\} \oplus^\perp \mathrm{Vect}\{\vec{\omega}_e\}^\perp$ with $\vec{0} \neq \vec{\omega}_e \times_e \overrightarrow{C_0B}$, thus $\vec{u}(B) \nparallel \vec{\omega}_e$, hence $B \notin \mathrm{Ax}(\vec{u})$. Thus $\mathrm{Ax}(\vec{u}) = C_0 + \mathrm{Vect}\{\vec{\omega}_e\}$.

2- $\vec{u}(C_0 + \lambda\vec{\omega}_e) = \vec{u}(C_0) + \vec{\omega}_e \times_e (\lambda\vec{\omega}_e) = \vec{u}(C_0) + \vec{0}$, thus $\vec{u}(C) = \vec{u}(C_0)$ for all $C \in C_0 + \mathrm{Vect}\{\vec{\omega}_e\}$.

3- If $B \notin C_0 + \mathrm{Vect}\{\vec{\omega}_e\}$ then $||\vec{u}(B)||_e^2 = ||\vec{u}(C_0) + \vec{\omega}_e \times_e \overrightarrow{C_0B}||_e^2 > ||\vec{u}(C_0)||_e^2$ (Pythagoras since $\vec{u}(C_0) \parallel \vec{\omega}_e$ is orthogonal to $\vec{\omega}_e \times_e \overrightarrow{C_0B}$).

4- $\vec{u}(B) \stackrel{(4.6)}{=} \vec{u}(C_0) + \vec{\omega}_e \times_e \overrightarrow{C_0B}$ with $\vec{u}(C_0) \parallel \vec{\omega}_e$ and $\vec{\omega}_e \times_e \overrightarrow{C_0B} \perp \vec{\omega}_e$. ∎

Exercise 4.12 How was the point C_0 in (4.11) found?

Answer. If $\vec{u}(O) \parallel \vec{\omega}_e$ then take $C_0 = O$. Else a drawing encourages to look for a $C_0 = O + \alpha\vec{\omega}_e \times_e \vec{u}(O)$ for some $\alpha \in \mathbb{R}$ because $\overrightarrow{OC_0}$ is then orthogonal to $\mathrm{Vect}\{\vec{\omega}_e\}$. Which gives $\vec{u}(C_0) = \vec{u}(O) + \vec{\omega}_e \times_e \overrightarrow{OC_0} = \vec{u}(O) + \vec{\omega}_e \times_e (\alpha\vec{\omega}_e \times_e \vec{u}(O)) = \vec{u}(O) + \alpha(\vec{\omega}_e \bullet_e \vec{u}(O))\vec{\omega}_e - \alpha||\vec{\omega}_e||^2\vec{u}(O)$. Hence we choose $\alpha = \frac{1}{||\vec{\omega}_e||^2}$: We get $\vec{u}(C_0) = \frac{1}{||\vec{\omega}_e||^2}(\vec{\omega}_e \bullet_e \vec{u}(O))\vec{\omega}_e$ parallel to $\vec{\omega}_e$, thus C_0 is in $\mathrm{Ax}(\vec{u})$: We have obtained (4.11). ∎

Exercise 4.13 Let $\vec{u}_1$ and $\vec{u}_2$ be two non constant screws s.t. $\vec{\omega}_{e1} + \vec{\omega}_{e2} \neq 0$. Find the axis of $\vec{u} := \vec{u}_1{+}\vec{u}_2$.

Answer. $\vec{u}_1(B) = \vec{u}_1(O) + \vec{\omega}_{e1} \times_e \overrightarrow{OB}$ and $\vec{u}_2(B) = \vec{u}_2(O) + \vec{\omega}_{e2} \times_e \overrightarrow{OB}$ give $(\vec{u}_1{+}\vec{u}_2)(B) = (\vec{u}_1(O) + \vec{u}_2(O)) + (\vec{\omega}_1{+}\vec{\omega}_2) \times_e \overrightarrow{OB}$. Thus $\mathrm{Ax}(\vec{u}_1{+}\vec{u}_2) = C + \mathrm{Vect}\{\vec{\omega}_1{+}\vec{\omega}_2\}$ where $C : \stackrel{(4.11)}{=} O + \dfrac{1}{||\vec{\omega}_1{+}\vec{\omega}_2||^2}(\vec{\omega}_1{+}\vec{\omega}_2) \times_e \vec{u}(O)$. ∎

4.5 The pitch

Let $\vec{u}$ be a non constant screw, i.e. $\vec{u}(B) = \vec{u}(A) + \vec{\omega}_e \times_e \overrightarrow{AB}$ for all A, B with $\vec{\omega}_e \neq \vec{0}$.

Definition 4.14 The pitch of a $\vec{u}$ is the real $h \in \mathbb{R}$ s.t., for any $C \in \text{Ax}(\vec{u})$,

$$\vec{u}(C) = h\vec{\omega}_e, \quad \text{i.e.} \quad h = \frac{\vec{u}(C) \bullet_e \vec{\omega}_e}{\omega_e^2}. \tag{4.14}$$

5 Twist = kinematic torsor = distributor

5.1 Definition

Definition 5.1 A twist (or kinematic screw or distributor)[1] is the name of the screw "The Eulerian velocity field of a rigid body".

Details: Let *Obj* be a rigid body, P_{Obj} its particles, $\widetilde{\Phi} : \left\{ \begin{array}{ll} [t_0, T] \times Obj & \to \mathbb{R}^3 \\ (t, P_{Obj}) & \to p(t) = \widetilde{\Phi}(t, P_{Obj}) \end{array} \right\}$ its motion (where $t_0, T \in \mathbb{R}$ and $t_0 < T$), and $Dom_t = \widetilde{\Phi}(t, Obj) \subset \mathbb{R}^3$ its position in $\mathbb{R}^3$ at t. Its Eulerian velocity field is the vector field $\vec{v} : \bigcup_{t\in[t_0,T]}(\{t\} \times Dom_t) \to \overrightarrow{\mathbb{R}^3}$ defined by $\vec{v}(t, p(t)) := \frac{\partial\widetilde{\Phi}}{\partial t}(t, P_{Obj})$ when $p(t) = \widetilde{\Phi}(t, P_{Obj})$.

Fix t and let $\vec{v}(t, p(t)) =^{\text{noted}} \vec{v}(p)$. Consider a Euclidean basis $(\vec{e}_i)$ and the associated Euclidean dot product $(\cdot,\cdot)_e$. The body being rigid, $\vec{v}$ is affine and antisymmetric (is a screw): $\vec{v}(q) = \vec{v}(p) + d\vec{v}(p).\overrightarrow{pq}$ with $d\vec{v}(p)$ independent of p and $d\vec{v} + d\vec{v}^T = 0$. So, with $\vec{\omega}_e \in \overrightarrow{\mathbb{R}^3}$ given by $d\vec{v}.() = \vec{\omega}_e \times_e ()$, for all $p, q \in Dom_t$,

$$\vec{v}(q) = \vec{v}(p) + \vec{\omega}_e \times_e \overrightarrow{pq}. \tag{5.1}$$

$\vec{\omega}_e$ is the vector angular velocity, and $\omega_e := ||\vec{\omega}_e||_e$ is the angular velocity.

Then artificially extending the body to infinity with zero density: $\text{Ax}(\vec{v}) = \{c \in \mathbb{R}^3 : \vec{v}(c) \parallel \vec{\omega}_e\}$ is well defined, and, with $c \in \text{Ax}(\vec{v})$ and $q \in Dom_t$, $\vec{v}(q) = \vec{v}(c) + \vec{\omega}_e \times_e \overrightarrow{cq}$ is an orthogonal decomposition of $\vec{v}(q)$ in $\text{Vect}\{\vec{\omega}_e\} \oplus^\perp \text{Vect}\{\vec{\omega}_e\}^\perp$.

Exercise 5.2 (5.1) gives the "equiprojectivity property": $\vec{v}(p).\overrightarrow{pq} = \vec{v}(q).\overrightarrow{pq}$. Prove it starting from $||\overrightarrow{p(t)q(t)}||_e =$ constant (rigid body) for all particles $P_{Obj}, Q_{Obj} \in Obj$ where $p(t) = \widetilde{\Phi}(t, P_{Obj})$ and $q(t) = \widetilde{\Phi}(t, Q_{Obj})$.

Answer. Choose a $O \in \mathbb{R}^3$. let $p(t) = \widetilde{\Phi}(t, P_{Obj})$ and $q(t) = \widetilde{\Phi}(t, Q_{Obj})$. Thus $\frac{d}{dt}\overrightarrow{p(t)q(t)} = \frac{d}{dt}\overrightarrow{Oq(t)} - \frac{d}{dt}\overrightarrow{Op(t)} = \vec{v}(t, q(t)) - \vec{v}(t, p(t))$. And $||\overrightarrow{p(t)q(t)}||_e^2 = (\overrightarrow{p(t)q(t)}, \overrightarrow{p(t)q(t)})_e =$ constant, thus $\frac{d}{dt}(\overrightarrow{p(t)q(t)}, \overrightarrow{p(t)q(t)})_e = 0 = 2(\frac{d}{dt}\overrightarrow{p(t)q(t)}, \overrightarrow{p(t)q(t)})_e$, thus $0 = (\vec{v}(t, q(t)) - \vec{v}(t, p(t))), \overrightarrow{p(t)q(t)})_e$ (equiprojectivity property). ∎

5.2 Pure rotation

Definition 5.3 A pure rotation is a non constant twist $\vec{v}$ s.t. $\exists c_0 \in \mathbb{R}^3$, $\vec{v}(c_0) = \vec{0}$; I.e.

$$\exists c_0 \in \mathbb{R}^3,\ \forall q \in \mathbb{R}^3,\ \vec{v}(q) = \vec{\omega}_e \times_e \overrightarrow{c_0q} \quad \text{with} \quad \vec{\omega}_e \neq \vec{0}. \tag{5.2}$$

(In which case $\vec{v}(q) \perp \vec{\omega}_e$ for all $q \in \mathbb{R}^3$ and $\text{Ax}(\vec{v}) = c_0 + \text{Vect}\{\vec{\omega}_e\}$).

Exercise 5.4 Fix $(\vec{e}_i)$, write $\times_e = \times$ and $\vec{\omega}_e = \vec{\omega}$, let $\vec{v}_1(q) = \vec{\omega}_1 \times \overrightarrow{c_1q}$ and $\vec{v}_2(q) = \vec{\omega}_2 \times \overrightarrow{c_2q}$.

1- Suppose $\text{Ax}(\vec{v}_1) \parallel \text{Ax}(\vec{v}_2)$, axes disjoint, and $\vec{\omega}_1+\vec{\omega}_2 \neq \vec{0}$. Find $\text{Ax}(\vec{v}_1+\vec{v}_2)$ and prove that $\vec{v}_1+\vec{v}_2$ is a pure rotation.

1'- Suppose $\text{Ax}(\vec{v}_1) \parallel \text{Ax}(\vec{v}_2)$, axes disjoint, and $\vec{\omega}_1+\vec{\omega}_2 = \vec{0}$. Prove that $\vec{v}_1+\vec{v}_2$ is a translation.

2- Suppose $\text{Ax}(\vec{v}_1) \nparallel \text{Ax}(\vec{v}_2)$ and the axes intersect at only one point O. Find $\text{Ax}(\vec{v}_1+\vec{v}_2)$, and prove that $\vec{v}_1+\vec{v}_2$ is a pure rotation.

3- Suppose $\text{Ax}(\vec{v}_1) \nparallel \text{Ax}(\vec{v}_2)$ and the axes don't intersect. Find $\text{Ax}(\vec{v}_1+\vec{v}_2)$, and prove that $\vec{v}_1+\vec{v}_2$ is not a pure rotation. Give a "simple" particular $c_0 \in \text{Ax}(\vec{v}_1+\vec{v}_2)$.

Answer. The notations tells: $c_1 \in \text{Ax}(\vec{v}_1)$, $c_2 \in \text{Ax}(\vec{v}_2)$, $(\vec{v}_1+\vec{v}_2)(q) = \vec{\omega}_1 \times \overrightarrow{c_1q} + \vec{\omega}_2 \times \overrightarrow{c_2q}$ for all q.

[1]Definition of a twist by R.S. Ball [1]: "A body is said to receive a twist about a screw when it is rotated about the screw, while it is at the same time translated parallel to the screw, through a distance equal to the product of the pitch and the circular measure of the angle of rotation; hence, *the canonical form to which the displacement of a rigid body can be reduced is a twist about a screw.*"

1- Here $\vec{\omega}_2 = \lambda\vec{\omega}_1$ with $\lambda \neq -1$, thus $(\vec{v}_1+\vec{v}_2)(q) = \vec{\omega}_1 \times (\overrightarrow{c_1q} + \lambda\overrightarrow{c_2q}) = (\lambda+1)\vec{\omega}_1 \times (\frac{1}{\lambda+1}\overrightarrow{c_1q} + \frac{\lambda}{\lambda+1}\overrightarrow{c_2q})$. Hence choose $c_0 \in \mathbb{R}^3$ s.t. $\frac{1}{\lambda+1}\overrightarrow{c_1c_0} + \frac{\lambda}{\lambda+1}\overrightarrow{c_2c_0} = \vec{0}$ (barycentric point on the straight line containing c_1 and c_2): We get $\vec{v}(c_0) = \vec{0}$ and $\mathrm{Ax}(\vec{v}_1+\vec{v}_2) = c_0 + \mathrm{Vect}\{\vec{\omega}_1+\vec{\omega}_2\}$. Remark (on barycentric points): We have $\overrightarrow{c_1c_0} = \frac{1}{\lambda+1}\overrightarrow{c_1c_2}$, thus c_0 in between c_1 and c_2 iff $0 < \frac{1}{\lambda+1} < 1$, i.e. iff $\lambda > 0$, i.e. iff $\vec{\omega}_1$ and $\vec{\omega}_2$ have the same orientation.

1'- $(\vec{v}_1+\vec{v}_2)(q) = (\vec{v}_1+\vec{v}_2)(p) + (\vec{\omega}_1+\vec{\omega}_2) \times \overrightarrow{pq} = (\vec{v}_1+\vec{v}_2)(p) + \vec{0}$ for all p,q, so $\vec{v}_1+\vec{v}_2$ is constant; Suppose $\exists q \in \mathbb{R}^3$ s.t. $(\vec{v}_1+\vec{v}_2)(q) = \vec{0}$: Hence $\vec{\omega}_1 \times \overrightarrow{c_1q} + (-\vec{\omega}_1) \times \overrightarrow{c_2q} = \vec{0}$, thus $\vec{\omega}_1 \times \overrightarrow{c_1c_2} = \vec{0}$, thus $\vec{\omega}_1 \parallel \overrightarrow{c_1c_2}$, absurd because the axes are parallel and disjoint. Thus $\vec{v}_1+\vec{v}_2 \neq \vec{0}$.

2- Take c_1=c_2=0, thus $(\vec{v}_1+\vec{v}_2)(q) = (\vec{\omega}_1+\vec{\omega}_2) \times \overrightarrow{Oq}$, thus $(\vec{v}_1+\vec{v}_2)(O) = \vec{0}$ and $\mathrm{Ax}(\vec{v}_1+\vec{v}_2) = O + \mathrm{Vect}\{\vec{\omega}_1+\vec{\omega}_2\}$.

3- Here $\vec{\omega} := \vec{\omega}_1+\vec{\omega}_2 \neq \vec{0}$ and (4.11) tells that c_0 defined by $\overrightarrow{c_1c_0} = \frac{1}{||\vec{\omega}||^2}\vec{\omega} \times (\vec{v}_1+\vec{v}_2)(c_1) = \frac{1}{||\vec{\omega}||^2}\vec{\omega} \times \vec{v}_2(c_1) = \frac{1}{||\vec{\omega}||^2}\vec{\omega} \times (\vec{\omega}_2 \times \overrightarrow{c_2c_1})$, i.e.

$$\overrightarrow{c_1c_0} = \frac{1}{||\vec{\omega}||^2}\Big((\vec{\omega} \bullet_e \overrightarrow{c_2c_1})\vec{\omega}_2 - (\vec{\omega} \bullet_e \vec{\omega}_2)\overrightarrow{c_2c_1}\Big) \tag{5.3}$$

is in $\mathrm{Ax}(\vec{v}_1+\vec{v}_2)$, so $\mathrm{Ax}(\vec{v}_1+\vec{v}_2) = c_0 + \mathrm{Vect}\{\vec{\omega}_1+\vec{\omega}_2\}$.

In particular, choose c_1 and c_2 s.t. $\overrightarrow{c_1c_2} \perp \vec{\omega}_1$ and $\perp \vec{\omega}_2$, i.e. the segment $[c_1, c_2]$ is the shortest segment joining $\mathrm{Ax}(\vec{v}_1)$ and $\mathrm{Ax}(\vec{v}_2)$. Thus $\overrightarrow{c_1c_2} \in \mathrm{Vect}\{\vec{\omega}_1, \vec{\omega}_2\}^\perp$ and $\overrightarrow{c_1c_2} \perp \vec{\omega}_1+\vec{\omega}_2$. Thus

$$\overrightarrow{c_1c_0} = -\frac{\vec{\omega} \bullet_e \vec{\omega}_2}{||\vec{\omega}||^2}\overrightarrow{c_2c_1}, \quad \text{and} \quad \overrightarrow{c_2c_0} = \overrightarrow{c_2c_1} + \overrightarrow{c_1c_0} = (1 - \frac{\vec{\omega} \bullet_e \vec{\omega}_2}{||\vec{\omega}||^2})\overrightarrow{c_2c_1}. \tag{5.4}$$

In particular c_0 is in the straight line containing c_1, c_2. Thus $\vec{v}_1(c_0) = \vec{\omega}_1 \times \overrightarrow{c_1c_0} = -\frac{\vec{\omega} \bullet_e \vec{\omega}_2}{||\vec{\omega}||^2}\vec{\omega}_1 \times \overrightarrow{c_2c_1}$, and $\vec{v}_2(c_0) = \vec{\omega}_2 \times \overrightarrow{c_2c_0} = (1 - \frac{\vec{\omega} \bullet_e \vec{\omega}_2}{||\vec{\omega}||^2})\vec{\omega}_2 \times \overrightarrow{c_2c_1}$. Thus $(\vec{v}_1+\vec{v}_2)(c_0) = (-\frac{\vec{\omega} \bullet_e \vec{\omega}_2}{||\vec{\omega}||^2}\vec{\omega}_1 + (1 - \frac{\vec{\omega} \bullet_e \vec{\omega}_2}{||\vec{\omega}||^2})\vec{\omega}_2) \times \overrightarrow{c_2c_1}$. And $\vec{\omega}_1$ and $\vec{\omega}_2$ are independent, thus $\vec{\omega}$ and $\vec{\omega}_2$ are independent, thus $\vec{\omega} \bullet_e \vec{\omega}_2 \neq 0$ and $(-\frac{\vec{\omega} \bullet_e \vec{\omega}_2}{||\vec{\omega}||^2}\vec{\omega}_1 + (1 - \frac{\vec{\omega} \bullet_e \vec{\omega}_2}{||\vec{\omega}||^2})\vec{\omega}_2) \neq \vec{0}$, together with $(-\frac{\vec{\omega} \bullet_e \vec{\omega}_2}{||\vec{\omega}||^2}\vec{\omega}_1 + (1 - \frac{\vec{\omega} \bullet_e \vec{\omega}_2}{||\vec{\omega}||^2})\vec{\omega}_2) \perp \overrightarrow{c_2c_1} \neq \vec{0}$; Thus $(\vec{v}_1+\vec{v}_2)(c_0) \neq \vec{0}$, thus $\vec{v}_1+\vec{v}_2$ isn't a pure rotation. ▪

Exercise 5.5 Prove: A twist $\vec{v}$ is the sum of a pure rotation and a translation.

Answer. With $\vec{v}(p) = \vec{v}(O) + \vec{\omega}_e \times_e \overrightarrow{Op}$: Call $\vec{v}_r$ the pure rotation defined by $\vec{v}_r(p) = \vec{\omega}_e \times_e \overrightarrow{Op}$ and call $\vec{v}_t$ the translation defined by $\vec{v}_t(p) = \vec{v}(O)$. We have $(\vec{v}_t + \vec{v}_r)(p) = \vec{v}(p)$, for all p, hence $\vec{v} = \vec{v}_r + \vec{v}_t$. ▪

6 Wrench = static torsor

6.1 Definition

Definition 6.1 A wrench is the name given to a screw $\vec{u}$ when, at some P_0, $\vec{u}$ is the moment of a force:

$$\vec{u}(P_0) = \vec{f}(P_f) \times_e \overrightarrow{P_fP_0} \quad (= \overrightarrow{P_0P_f} \times_e \vec{f}(P_f)) \quad \in \mathrm{Vect}\{\vec{f}(P_f), \overrightarrow{P_fP_0}\}^\perp, \tag{6.1}$$

where $\vec{f}(P_f)$ is the vector force applied at P_f.

And the "moment arm" at P_0 is the distance between the straight line $P_f + \mathrm{Vect}\{\vec{f}(P_f)\}$ and P_0, i.e. the distance between P_0 and its orthogonal projection on $P_f + \mathrm{Vect}\{\vec{f}(P_f)\}$. Drawing.

This definition supposes that the domain of definition of $\vec{u}$ is $Dom = \{P_0\}$.

First generalization: Dom can be extended to the segment $[P_0, P_f] = \{P \in \mathbb{R}^3 : \overrightarrow{P_0P} = \alpha\overrightarrow{P_0P_f},\ \alpha \in [0,1]\}$, corresponding to the position of a rigid body like "the wheel nut at P_0 welded to the wrench used to unscrew it". Thus, for all $P \in [P_0, P_f]$,

$$\vec{u}(P) = \vec{f}(P_f) \times_e \overrightarrow{P_fP} \quad (\in \mathrm{Vect}\{\vec{f}(P_f), \overrightarrow{P_fP}\}^\perp). \tag{6.2}$$

Here $\vec{u}(P_f) = \vec{0}$ (the moment arm vanishes).

Second generalization: $\vec{u}$ can be extended to $\mathbb{R}^3$ (with $\vec{u}$ supposed affine antisymmetric). Then the resultant is $\vec{f}(P_f)$, and the axis is $P_f + \mathrm{Vect}\{\vec{f}(P_f)\}$

6.2 Couple of forces and resulting wrench

Consider two wrenches given by at P_0 by $\vec{u}_1(P_0) = \vec{f}_1(P_{f_1}) \times_e \overrightarrow{P_{f_1}P_0}$ and $\vec{u}_2(P_{f_2}) = \vec{f}_2(P_{f_2}) \times_e \overrightarrow{P_{f_2}P_0}$. Thus, at P_0,

$$(\vec{u}_1+\vec{u}_2)(P_0) = \vec{f}_1(P_{f_1}) \times_e \overrightarrow{P_{f_1}P_0} + \vec{f}_2(P_{f_2}) \times_e \overrightarrow{P_{f_2}P_0} \overset{\text{noted}}{=} \vec{u}(P_0). \tag{6.3}$$

A fundamental example: Suppose that $\vec{f_2}(P_{f_2}) = -\vec{f_1}(P_{f_1})$ and $\overrightarrow{P_{f_1}P_0} = -\overrightarrow{P_{f_2}P_0}$ and $\vec{f_1}(P_{f_1}) \perp \overrightarrow{P_{f_1}P_0}$ (drawing: P_0 is the position of a nut holding a car wheel and P_{f_1} and P_{f_2} are the ends of a lug wrench used to unscrew the nut). We get "the couple at P_0" (expected result, drawing):

$$\vec{u}(P_0) = 2\vec{f_1}(P_{f_1}) \times_e \overrightarrow{P_{f_1}P_0} = \vec{f_1}(P_{f_1}) \times_e (2\overrightarrow{P_{f_1}P_0}) \quad (= \vec{f_1}(P_{f_1}) \times_e \overrightarrow{P_{f_1}P_{f_2}}). \tag{6.4}$$

First generalization: *Dom* can be extended to the segment $[P_0, P_f]$; We get, at any $P \in [P_{f_1}, P_{f_2}]$,

$$\vec{u}(P) = \vec{f_1}(P_{f_1}) \times_e \overrightarrow{P_{f_1}P} - \vec{f_1}(P_{f_1}) \times_e \overrightarrow{P_{f_2}P} = \vec{f_1}(P_{f_1}) \times_e \overrightarrow{P_{f_1}P_{f_2}} = \overrightarrow{\text{constant}}. \tag{6.5}$$

It is independent of P: Indeed the "moment arms" $d(P, P_{f_1})$ and $d(P, P_{f_2})$ ("one short and one long") give (6.5). Thus the screw $\vec{u}$ is constant along $[P_{f_1}, P_{f_2}]$.

Second generalization: *Dom* can be extended to $\mathbb{R}^3$: The screw $\vec{u}$ is constant in $\mathbb{R}^3$.

References

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