Des cascades infiniment divisibles multifractales.

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Multifractal Processes and Scale Invariance

$\delta_l X(t) = X(t+l) - X(t)$: Increment at time t, scale l.

$M(q,l) = E\left(|\delta_l X(t)|^q\right) \sim K_q l^{\zeta_q}$.
$\Longrightarrow$ "Scale invariance" of $\delta_l X(t)$
ζ_q : Multifractal exponents

- Self-similar (Monofractal) Processes

  $\zeta_q = qH$ is linear, H : Hurst exponent.

  Brownian motion : $H = 1/2$

  Levy walk (α-stable process) : $H = 1/\alpha$,

- Multifractal Processes

  ζ_q is non linear

  • • •

$\Longrightarrow$ Existence of an "integral" scale $l = L$.

$\Longrightarrow$ $M(q,l) \simeq K_q l^{\zeta_q}$, $\forall l \leq L$ (ζ_q convex).

Multi/Mono fractal Processes

- Self Similar processes ($\delta_l X(t) = X(t+l) - X(t)$)

$$\{\delta_{\lambda l} X(\lambda t)\}_t =^{law} \lambda^H \{\delta_l X(t)\}_t$$

$\Longrightarrow$ a single exponent H

$\Longrightarrow$ $\zeta_q = qH$, **monofractal** processes

- **Multifractality** (*Castaing 90*) $\simeq$ H is stochastic

$$\delta_{\lambda l} X(\lambda t) =^{law} W_\lambda \delta_l X(t), \qquad (t \leq L)$$

W_λ : log-infinitely divisible, independant of X

G : law of H

- $\zeta_q = \ln \hat{G}(-iq) \Longrightarrow$ non-linear

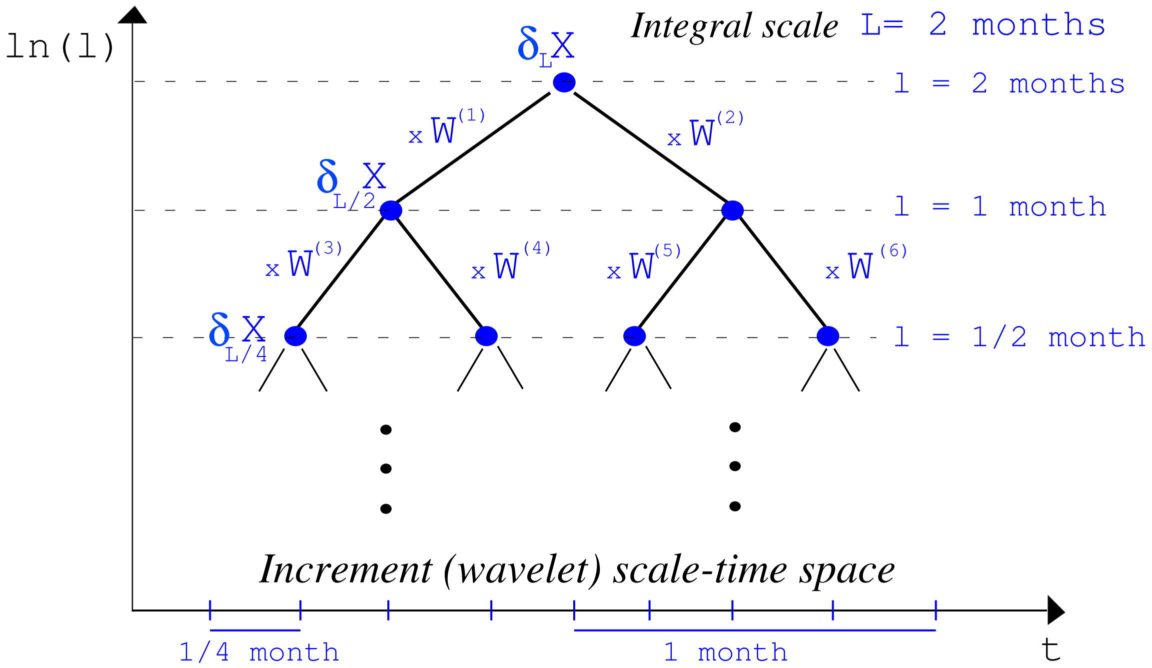
- $P_{\lambda l}(x) = \int du e^{-u} P_l(e^{-u}x) G^{\ln \lambda}(u)$

$P_l(x)$: pdf of $\delta_l X(t)$

$\Longrightarrow$ Shape of pdf changes across scales.

Cascade processes

$$\{\delta_{\lambda l} X(\lambda t)\}_t = \lambda^H \{\delta_l X(t)\}_t = W_\lambda \{\delta_l X(t)\}_t$$



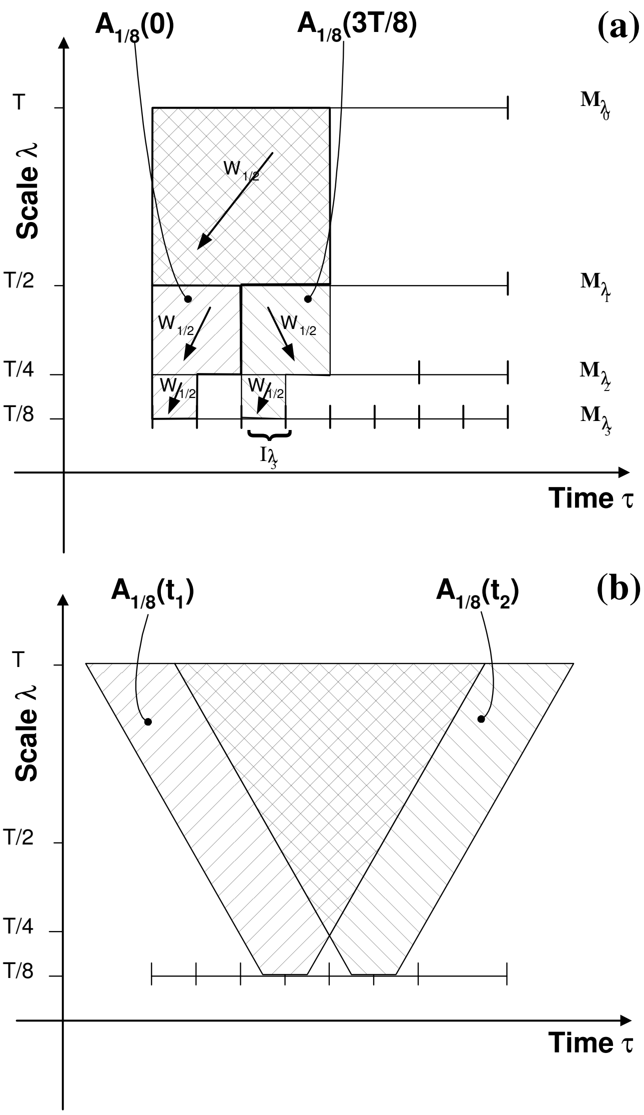
$\Longrightarrow$ Correlation structure

Magnitude : $\omega_l(t) = \frac{1}{2} \ln |\delta_l X(t)|^2$

$$Cov(\omega_l(t), \omega_l(t+\tau)) \simeq -C \ln(\tau/L), \quad l << \tau < L$$

- **Particular scale ratio (discrete scale invariance)**
- **Increments are non stationary**
- **Top-Bottom approach**

Infinitely divisible MRW



- P : infinit. div. stoch. 2D measure distributed with respect to $\mu(dt, dl) = dtdl/l^2$.

$E\left(e^{pP(\mathcal{A})}\right) = e^{\phi(-ip)\mu(\mathcal{A})}$,

(ϕ : Lévy Khintchine, $\phi(-i) = 0$)

- $M_l(I) = \int_I e^{\omega_l}(t)dt$ with $\omega_l(t) = P(\mathcal{A}_l(t))$
- $M(dt) = \lim_{l \to 0^+} M_l(dt)$

Main Results (Measures)

(Bacry, Muzy, 2002)

Existence - Non degeneracy - Moments

- with probability one, $M_l(dt) \rightarrow^{weakly} M(dt)$,
- $\zeta_q = q - \phi(-iq)$
- $\exists\, \epsilon > 0,\ \zeta_{1+\epsilon} > 1 \Rightarrow E\left(M([0,t])\right) = t.$
- $\zeta_q > 1 \text{``} \Longleftrightarrow \text{''} E\left(M([0,t])^q\right) < +\infty.$

Exact Scaling *(particular cone shape)*

- $\{\omega_{\lambda l}(\lambda t)\}_{t \leq L} =^{law} \Omega_\lambda + \{\omega_l(t)\}_{t \leq L}$,

where Ω_λ is inf. div. with $E\left(e^{iq\Omega_\lambda}\right) = \lambda^{-\varphi(q)}$.

- $\{M([0,\lambda t])\}_{t \leq L} =^{law} W_\lambda \{M([0,t])\}_{t \leq L}$,

with $W_\lambda = \lambda e^{\Omega_\lambda}$, for fixed $\lambda \in]01[$.

- $E\left(M([0,t])^q\right) = \left(\frac{t}{L}\right)^{\zeta_q} E\left(M[0,L])^q\right), \forall t \leq L$

Asymptotic Scaling *("generic" cone shape)*

- $E\left(M([0,t])^q\right) \sim_{t \rightarrow 0^+} \left(\frac{t}{L}\right)^{\zeta_q} E\left(M([0,L])^q\right).$

Main Results (Processes)

(Bacry, Muzy, 2002)

Definition (using Wiener noise $dW(u)$)

- $X(t) = \lim_{l \to 0^+} \int_0^t e^{\omega_l(u)/2} dW(u)$,

Exact Scaling *(particular cone shape)*

- $\zeta_q > 1 `` \Longleftrightarrow '' E\left(|X(t)|^{2q}\right) < +\infty$.
- $\{X(t)\}_{t \le L} \ =^{law} W_\lambda \{X(t)\}_{t \le L}, \quad \forall\, \lambda \in]0,1[$,
- $E\left(|X(t)|^{2q}\right) = \left(\frac{t}{L}\right)^{\zeta_q} E\left(|X(L)|^{2q}\right), \ \ \forall t \le L$.

Asymptotic Scaling *("generic" cone shape)*

- $E\left(|X(t)|^{2q}\right) \ \ \sim^{t \to 0^+} \ \ \left(\frac{t}{L}\right)^{\zeta_q} E\left(|X(L)|^{2q}\right)$.

Examples

- $P(dt, dl) = 0$

$\rightarrow \zeta_q = q.$

$\rightarrow$ M is the Lebesgue measure, (X is a Brownian motion)

- $P(dt, dl)$ is normal

$\rightarrow \zeta_q = q(1 + \frac{\lambda^2}{2}) - \frac{\lambda^2}{2} q^2.$

$\rightarrow$ Exact scaling case : Log normal MRW *(Bacry, Delour, Muzy 2000)*,

- $P(dt, dl)$ is (compound) Poisson

$\rightarrow \zeta_q = qm - C\,(E(W^q) - 1)$

$\rightarrow$ Asymptotic scaling case : Product of cylindrical pulses *(Barral, Mandelbrot 2000)*,

- $P(dt, dl)$ is (left-sided) α stable

$\rightarrow \zeta_q = qm - \sigma^\alpha |q|^\alpha$

- $P(dt, dl)$ is Gamma

$\rightarrow \zeta_q = qm - C\gamma^2 \ln(\frac{\gamma}{\gamma - q})$

The log-normal MRW model

(Bacry, Delour, Muzy 2000)

- Discrete time : Δt

$$X_{\Delta t}(t) = X_{\Delta t}(t - \Delta t) + e^{\Omega_{\Delta t}}(t)\epsilon_{\Delta t}(t)$$

  - $\Omega_{\Delta t}$: Gaussian stationary process

    independant of $\epsilon_{\Delta t}(t)$

    with autocorrelation $R_{\Delta t}(t)$

  - $\epsilon_{\Delta t}$: White Gaussian noise

    $< \epsilon_{\Delta t} >= 0, < \epsilon^2_{\Delta t} >= \sigma^2 \Delta t,$

- $\phi(-i) = 0 \Longleftrightarrow \; < \Omega_{\Delta t} >= -Var(\Omega_{\Delta t}) = R_{\Delta t}(0)$

$\Longrightarrow Var(X_{\Delta t}(t)) = \sigma^2 t$

- Exact scaling

$$\Longrightarrow \; R_{\Delta t}(t) = \begin{cases} \lambda^2 \ln\left(\frac{L}{(|t|+\Delta t)}\right) & \text{for } |t| \le L \\ 0 & \text{otherwise} \end{cases}$$

The log-normal MRW model

- Continuous scale invariance

  - $< |X(t)|^q >= K_q (l/L)^{\zeta_q}, \quad \forall t \leq L,$

    $K_q = L^{q/2} \sigma^q (q-1)!! \prod_{k=0}^{q/2-1} \frac{\Gamma(1-2\lambda^2 k)^2 \Gamma(1-2\lambda^2(k+1))}{\Gamma(2-2\lambda^2(q/2+k-1))\Gamma(1-2\lambda^2)}$

  - $\zeta_q = \ln \hat{G}(iq) = q/2 - q(q/2-1)\lambda^2$

- $< |X(t)|^q > \; = \; K'_q t^q, \qquad (t >> L)$

  $\simeq$ Brownian at large scale ($\zeta_q = q/2$)

- $\zeta_q < 1$ "$\Longleftrightarrow$" $< |X|^q >= +\infty$

  Fat tails

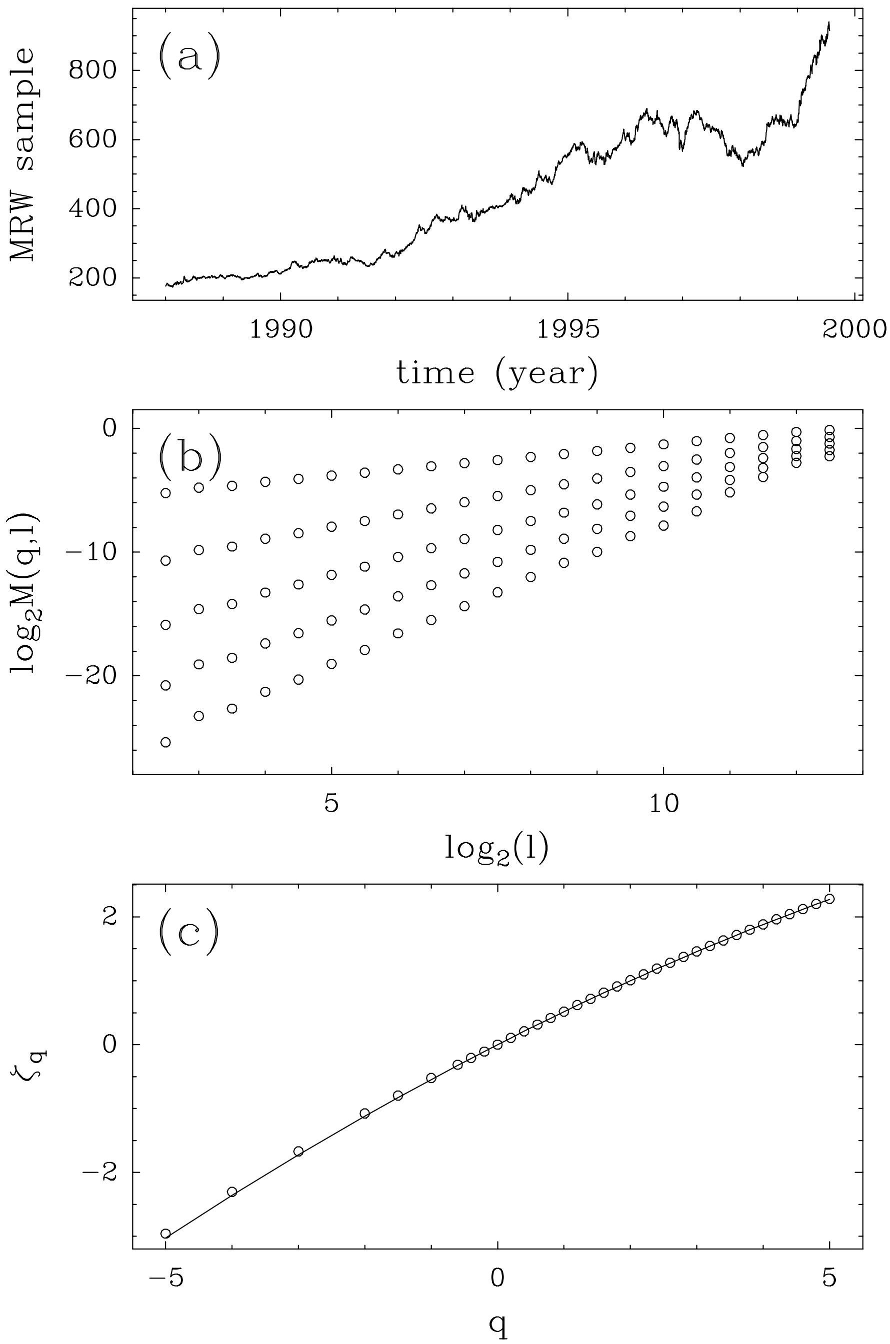
- $< |\delta_l X(t)|^{2p} |\delta_l X(t+\tau)|^{2p} > \; \simeq^{l << \tau} \; C_{2p} \left(\frac{\tau}{L}\right)^{-\lambda^2 p^2},$

  - "Magnitude log correlations" for $\lambda << 1$ (2004)

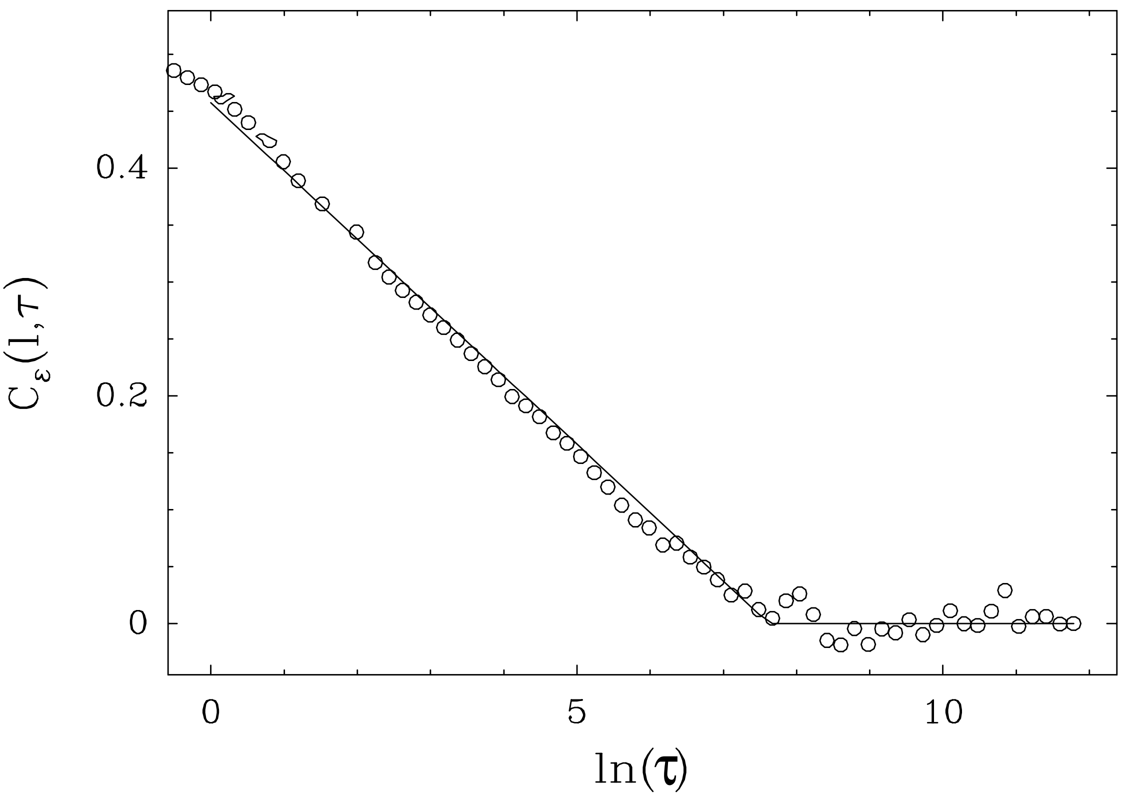
    $Cov(\omega_l(t), \omega_l(t+\tau)) \simeq -\lambda^2 \ln(\tau/L), \quad l < \tau < L$

  where $\omega_l(t) = \frac{1}{2} \ln |\delta_l X(t)|^2$

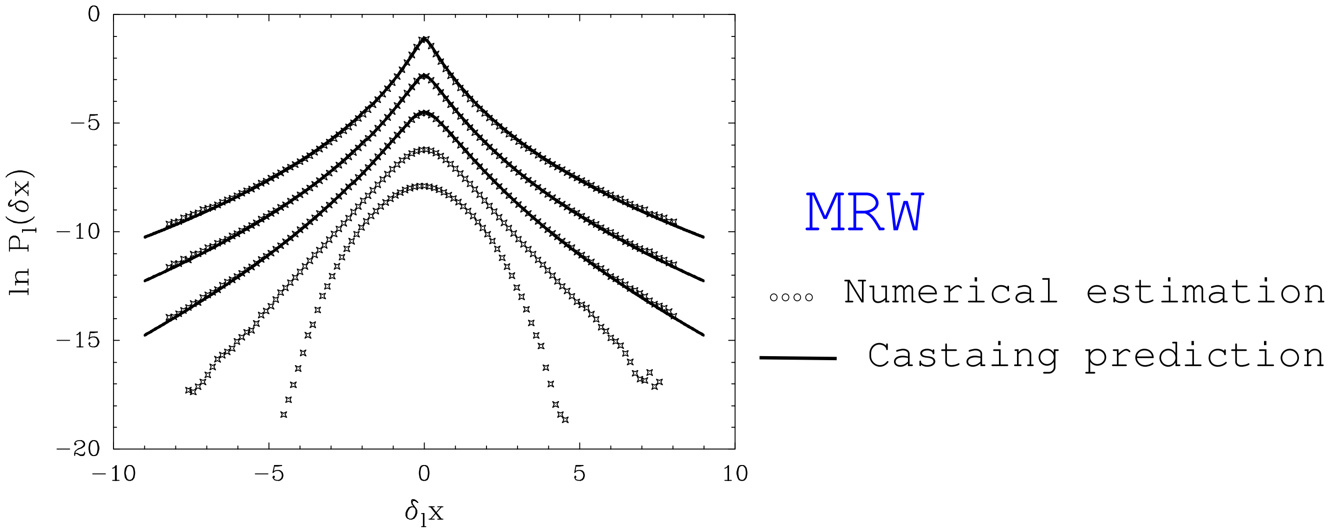
Multifractal analysis of a log-normal MRW

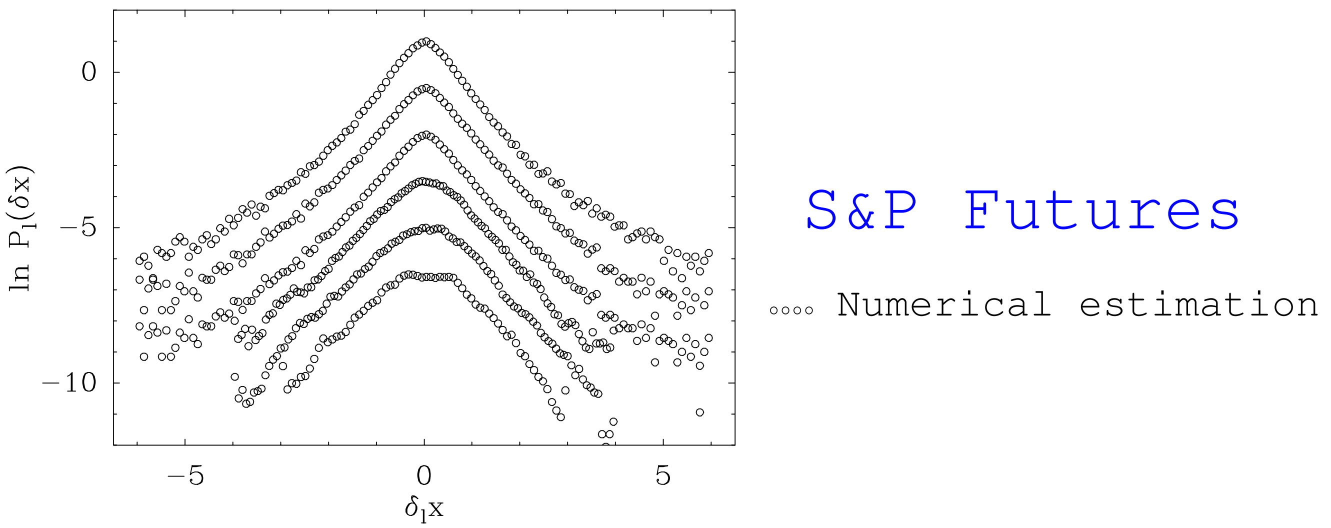


Magnitude correlation of a log-normal MRW



Continuous deformation of increment pdf's across scales





Parameter Estimation

(Bacry, Kojemiak, Muzy, 2004)

- Integral scale L

  $\longrightarrow$ decorrelation scale

- Variance σ^2

  $\longrightarrow$ variance of the increments

- Multifractal coefficient λ^2

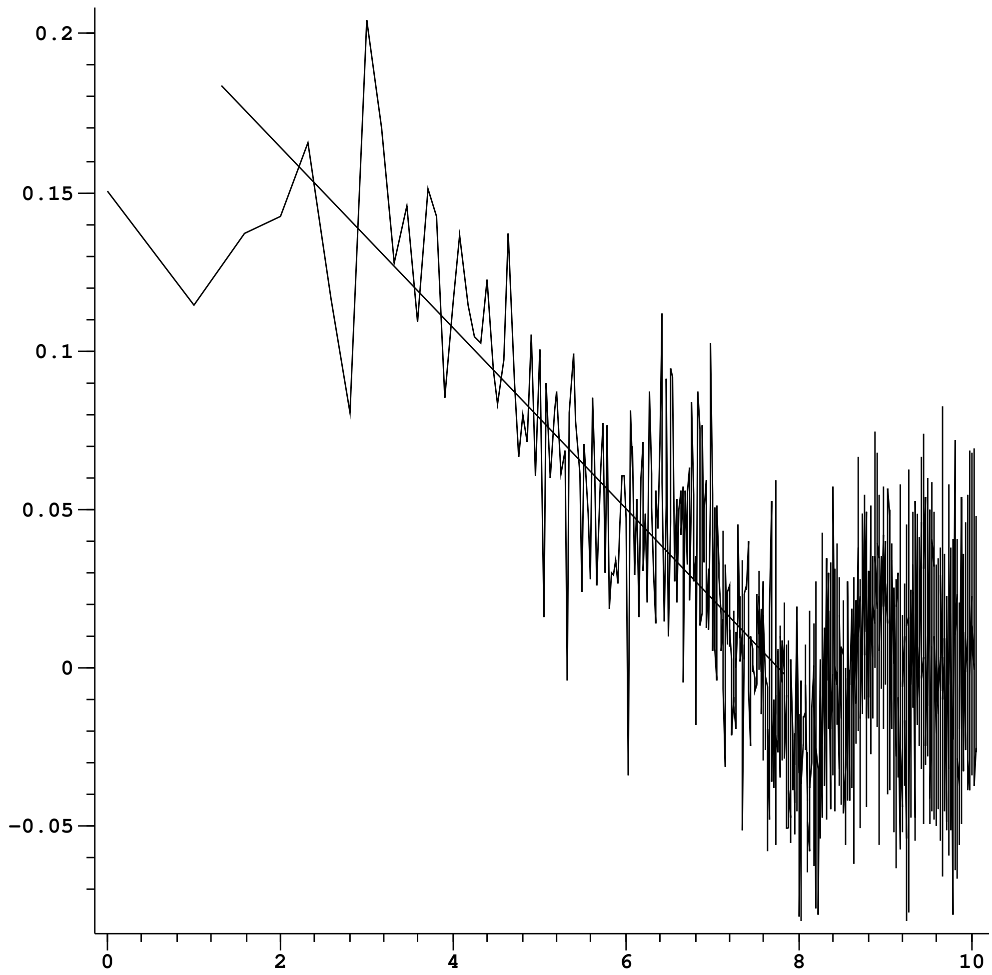
  $\longrightarrow$ correlation of the log of the increments $\omega_l(t)$

  $$Cov(\omega_l(t), \omega_l(t+\tau)) \simeq -\lambda^2 \ln(\tau/L), \quad l < \tau < L$$

$\Longrightarrow$ GMM estimation

Parameter Estimation (daily data)

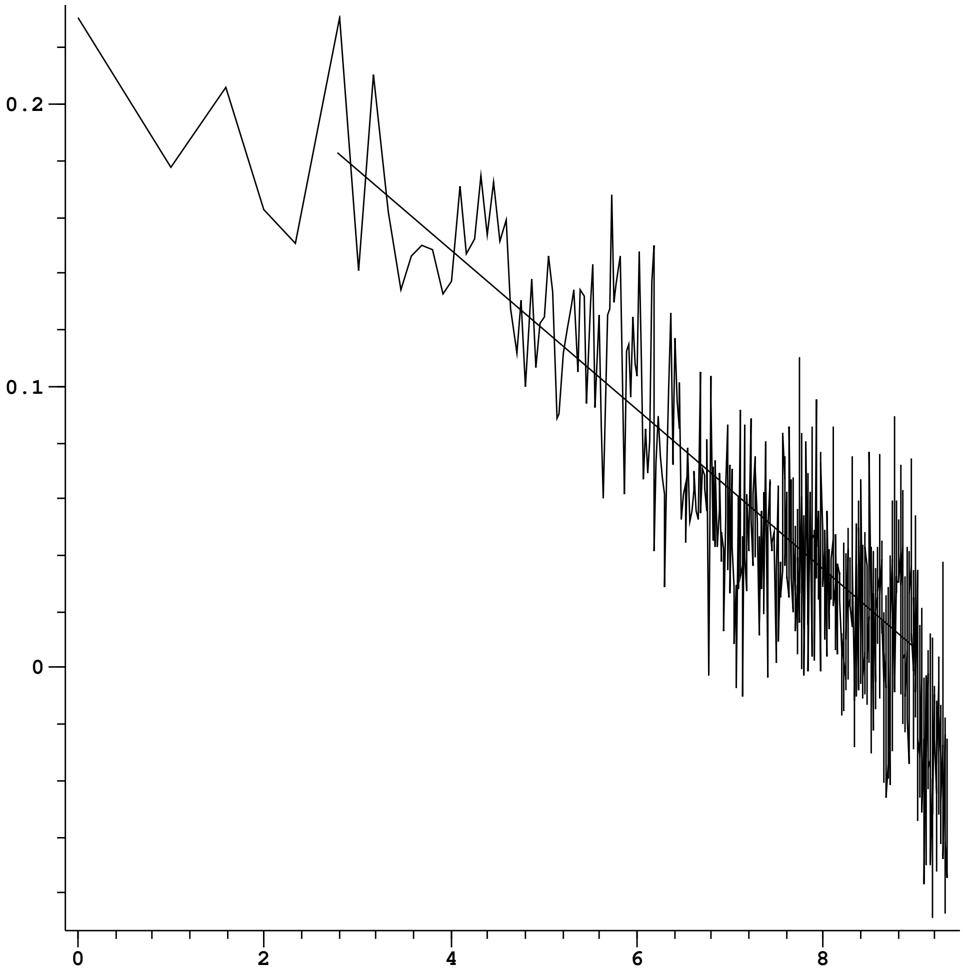
French Index daily data
1/1/1973 to 31/12/97 (6239 points)



$\lambda^2 = 0.029 \pm 0.01$, $L = 1$ year ($\in$ [6 months, 2 years])

Parameter Estimation (daily data)

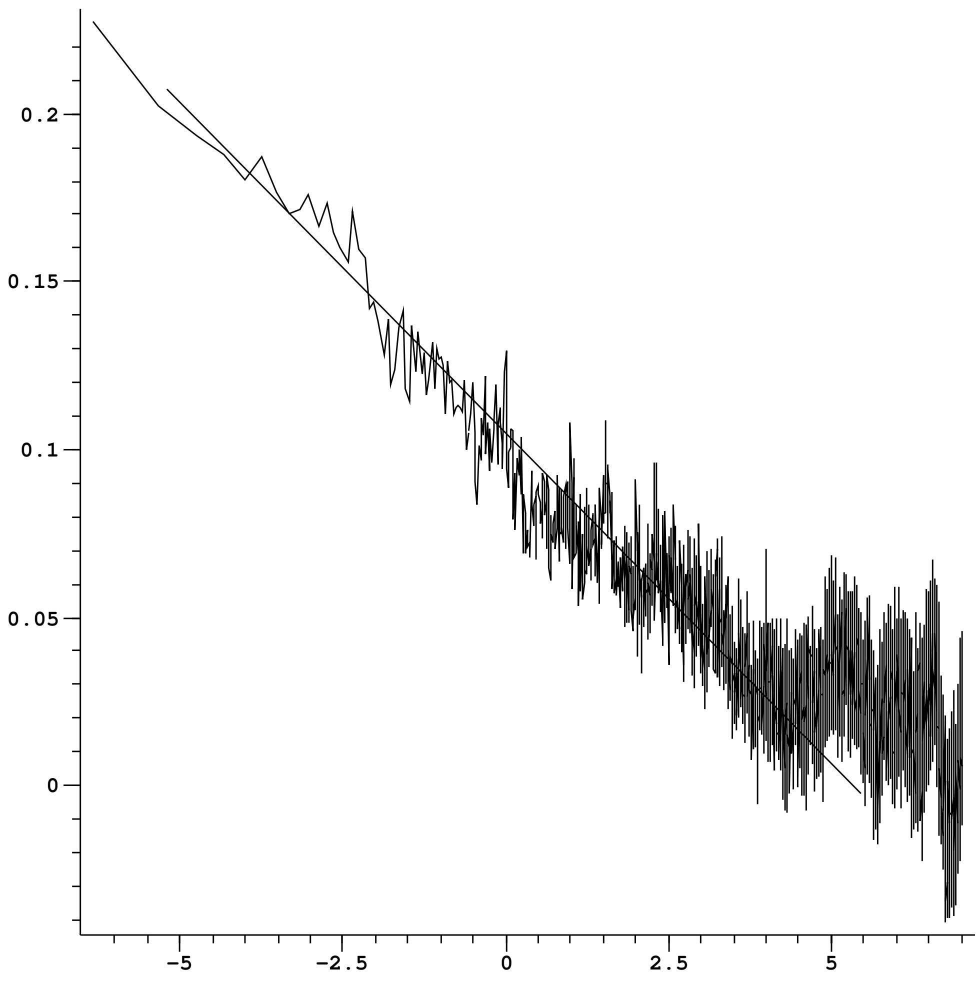
Italian Index daily data
1/1/1973 to 31/12/97 (6239 points)



$\lambda^2 = 0.029 \pm 0.015$, $L = 2$ years ($\in$ [1 year, 5 years])

Parameter Estimation (intraday)

S&P 500 intraday data (5mn ticks, 1996-1998)



$\lambda^2 \simeq 0.03$, $L \simeq 6$ months

Parameter Estimation

Series	Size	λ^2	L
S&P500 index	7.10^4	0.03	6 months
Future S&P500	7.10^4	0.025	3 years
FTSE100 index	7.10^4	0.028	1.2 year
Future FTSE100	7.10^4	0.029	1 year
Future JY/USD	7.10^4	0.02	6 months
Nikkei 225	7.10^4	0.030	1.5 year
Future Nikkei	7.10^4	0.02	6 months
Japanese Yen	7.10^4	0.022	1.0 year
French index	6.10^3	0.029	1 year
Italian index	6.10^3	0.029	2 years
Canadian index	6.10^3	0.032	1.5 years
German index	6.10^3	0.027	6 years
UK index	6.10^3	0.023	6 years
hong-kong index	6.10^3	0.037	6 years

Aggregation properties

• For $\lambda << 1$, analytic formula for p-points any order moments, volatility correlation structure across scales...

• ACTUALLY, for $\lambda << 1$, at any scale l

$$\delta_l X(nl) \simeq \epsilon_l[n] e^{\Omega_l[n]}$$

- $\epsilon_l[n]$ Gaussian white noise
- $e^{\Omega_l[n]}$ log-normal stochastic variance at scale l

$\Longrightarrow$ Variance prediction/estimation

Prospects

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Theory

log-normal (inf. div.) parameter estimation

Asymetry past/future (Leverage, HARCH effect,...)

Multivariate MRW (Log Normal)

Multifractal formalism

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Application (mainly finance)

(multivariate) Volatility prediction/estimation

(multivariate) Value at Risk (VaR) prediction/estimation

$\Longrightarrow$ non-gaussian dynamical portfolio management.

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