Isometric embeddings of subdivided complete graphs in the hypercube

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Why am I here?

(70’s) Djoković, and Graham and Pollack investigate partial cubes

('94) Chepoi-Tardif’s conjecture on partial cubes

('02) Negative answer by Brešar and Klavžar, using wheels

('03) Study of all subdivisions of wheels (Gravier et al.)

('06) High-density graphs with study of $K_5$ (Aïder et al.)

('07) Our work on subdivisions of $K_n$
Geodesics

- af-path
- af-geodesic
- not unique
Geodesics

- "af-path"
- "af-geodesic"
- "not unique"
Geodesics

- $af$-path
- $af$-geodesic
- not unique
Isometric embeddings

Geodesics

Previous work

Our Result

Future work

- af-path
- af-geodesic
- not unique
Isometric embedding

$H$ is isometrically embeddable by $\phi$ into $G$ if

$$d_H(u, v) = d_G(\phi(u), \phi(v))$$

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A partial cube is a graph isometrically embeddable in a hypercube.

Rem. : A partial cube is bipartite.
Subdivision of a graph

Given a graph $G$, a subdivision of it can be obtained by adding vertices on edges.

Isometric embeddings of subdivided complete graphs in the hypercube
Subdivision of a graph

Given $G$, $S(G)$ is the graph obtained by subdividing each edge once.

$S(K_4)$
Characterization

Two edges $e = xy$ and $f = uv$ are in Djoković-Winkler relation $\Theta$ if $d(x, u) + d(y, v) \neq d(x, v) + d(y, u)$.

- $C$ isometric even cycle
- $e, f$ opposite edges
- Then $e \Theta f$
Characterization

Two edges $e = xy$ and $f = uv$ are in Djoković-Winkler relation $\Theta$ if $d(x, u) + d(y, v) \neq d(x, v) + d(y, u)$.

**Theorem 1 [Winkler ’84]**
Given $G$ a bipartite graph, $G$ is a partial cube if and only if $\Theta = \Theta^*$

## Characterization

**Theorem 1 [Winkler ’84]**

Given $G$ a bipartite graph, $G$ is a partial cube if and only if $\Theta = \Theta^*$

- $\Rightarrow$ recognition algorithm in $O(mn)$ ($O(n^2)$ [Eppstein ’07])
- faster algorithms for specific graphs [Brešar et al. ’03]

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Structural characterizations

Wheels

**Theorem 2 [Gravier et al. ’03]**

Let $k \geq 3$ and $W$ a subdivision of $W_k$; $W$ is a partial cube if and only if, either its rays are not subdivided and each other edge is odd-subdivided, or it is isomorphic to $S(K_4)$.

Structural characterizations

**Theorem 2 [Gravier et al. ’03]**

Let $k \geq 3$ and $W$ a subdivision of $W_k$; $W$ is a partial cube if and only if, either its rays are not subdivided and each other edge is odd-subdivided, or it is isomorphic to $S(K_4)$.

$S(K_4)$ embedded in $Q_4$
Coming to cliques

Theorem 3 [Klavžar-Lipovec ’04]
For any $n \geq 1$, $S(K_n)$ is a partial cube.

Coming to cliques

**Proposition 4 [Aïder et al. ’06]**

Given $G$ a subdivision of $K_5$, $G$ is a partial cube if and only if, either $G \simeq S(K_5)$, or $G$ contains a universal vertex and other edges are odd-subdivided.

Coming to cliques

**Proposition 4 [Aïder et al. ’06]**

Given $G$ a subdivision of $K_5$, $G$ is a partial cube if and only if, either $G \simeq S(K_5)$, or $G$ contains a universal vertex and other edges are odd-subdivided.
Main result

**Theorem 5**
Given $G$ a subdivision of a complete graph $K_n$ ($n \geq 4$), $G$ is a partial cube if and only if, either $G$ is isomorphic to $S(K_n)$, or $G$ contains a universal vertex $u$ and other edges are odd-subdivided.
**Sketch of the proof**

**Proof.**

An $G'$ is isometric to $G$ if it is a partial cube, $G_0'$ is also a partial cube. With the induction hypothesis, there exists $x \in G'$ a universal vertex in $G'$ or $G_0'$ is isometric to $G(K_n)$. We then consider the case when $G(x) = \mathbb{H}(n)$. Previous work

- If $x \in G'(x)$, let us prove that $x$ is isometric to $G$. As $G$ is not isomorphic to $G(K_n)$, there exists $y \in G'(x)$ such that $G(y) = \mathbb{H}(n)$ is not a subgraph of $G$. Then, by Theorem 2, $G(x)$ is the only possible universal vertex in $G'(x)$, and $G(y)$ is a partial cube. Therefore, $G(x)$ is a universal vertex in $G'(x)$.

Our Result

- If there exists $z < G'(x)$ is universal in $G'(x)$, let $y \in G'(x)$, then $y \in G'(x)$ is universal in $G'(x)$.

Future work

- As $z$ is universal in $G'(x)$, we can assume that a shortest path from $y$ to $z$ contains $x, y'$. Therefore, $(x, y, z)$ is isometric to $G$. Let $x'$ be another vertex in $G'(x)$, and $y' \in G'(x)$ is a shortest path from $y$ to $z$ which is universal in $G'(x)$. Therefore, $G'(x)$ is a partial cube.

We can split vertices of $G'(x)$ into two non-empty sets $K$ containing the principal vertices, such that $G(y)$ is isometric to $G'(x)$ and $G(z)$ is the only universal vertex in $G'(x)$.

We prove that $G(x)$ is isometric to $G$. Let $G'$ be isometric to $G$. Then, by Theorem 2, $G(x)$ is a partial cube. Therefore, $G(x)$ is a universal vertex in $G'(x)$. We then consider the case when $G(x) = \mathbb{H}(n)$.
**Sketch of the proof**

\[ G \text{ partial cube} \iff \begin{cases} G \simeq S(K_n) \\ \text{universal vertex } u, \text{ even cycles} \end{cases} \]

**Sufficient condition**

- if \( G \simeq S(K_n) \), proven [Klavžar Lipovec ’04]

**Theorem 3 [Klavžar-Lipovec ’04]**

For any \( n \geq 1 \), \( S(K_n) \) is a partial cube.
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**Sufficient condition**

- if \( G \simeq S(K_n) \), proven [Klavžar Lipovec ’04]
- else,
  1. we embed the clique with a universal vertex and all other edges subdivided once [Aïder et al. ’06]
  2. for each edge we add the even number of missing subdivisions : dimension \( \uparrow \)
Sketch of the proof

![Diagram of isometric embeddings of subdivided complete graphs in the hypercube]
Sketch of the proof

Isometric embeddings of subdivided complete graphs in the hypercube
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\[ G \text{ partial cube} \iff \begin{cases} G \simeq S(K_n) \\ \text{universal vertex } u, \text{ even cycles} \end{cases} \]

**Necessary condition**

We proceed by induction:

- \( K_4 \) ['03] and \( K_5 \) ['06] have been treated before.
Sketch of the proof

$G$ partial cube $\iff \begin{cases} G \cong S(K_n) \\ \text{universal vertex } u, \text{ even cycles} \end{cases}$

**Necessary condition**

$G$ subdivision of $K_n$ :

- if $u$ principal (in $K_n$) vertex of $G$ such as $G \setminus u$ isometric
  - with induction hypothesis, the theorem is proven.
Sketch of the proof

\[ G \text{ partial cube} \iff \begin{cases} G \cong S(K_n) \\ \text{universal vertex } u, \text{ even cycles} \end{cases} \]

**Necessary condition**

Every vertex of \( G \) is isometrically needed. We choose \( u \) and partition the graph.
Sketch of the proof

(a) $\mathcal{L}$  
(b) $\mathcal{I}$  
(c) $\mathcal{C}$  
(d) $\Lambda$  
(e) $\mathcal{R}$

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Sketch of the proof

We then prove:

- No vertex of $K$ has type $C$
- Each vertex of $L$ sees exactly one vertex of $K$
- Consequently, $K$ is a singleton.
- Thus, $u$ could be isometrically removed.

Contradiction
Sketch of the proof

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Contradiction
Corollary

Let $G$ be a subdivision of a graph of order $n$ such as each edge is odd-subdivided. If we add $u$ a vertex adjacent to each principal vertex of $G$, the graph obtained is a partial cube.
Work to be done

- Subdivisions of cliques in products of $K_3$
- Subdivisions of cliques in Hamming graphs
- Graphs $\lambda$-scale embeddable in hypercubes
- Graphs $\lambda$-scale embeddable in Hamming graphs
Embeddings in power of $K_3$

**Theorem 6 [Beaudou-Gravier ’07]**

A subdivision of a clique $K_n$ ($n \geq 4$) is isometrically embeddable in a power of $K_3$ if and only if it is a partial cube.

**Key Lemma**

$G$ isometrically embeddable in a power of $K_3$. If $G$ is not bipartite, then it contains an induced $K_3$.

L. Beaudou and S. Gravier, *Isometric embeddings of subdivided cliques in products of $K_3*, To be submitted.*

Free at last...
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